Review 3

1. Decide whether the following statements are True (T) or False (F). On the inside cover of your blue book, list the letters (a) through (f) and put a T or F next to each one. **Answers only** will be graded.

a) By L’Hôpital’s Rule \( \lim_{{x \to 1}} \frac{3x^2 + 2x - 5}{2x^2 - 3} = \lim_{{x \to 1}} \frac{6x + 2}{4x} = 2. \)

b) If \( \lim_{{k \to \infty}} b_k = 0 \) then \( \sum_{{k=1}}^{\infty} b_k \) converges.

c) The series \( \frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \cdots \) converges.

d) The series \( \sum_{{k=1}}^{\infty} \frac{3^{k+1}}{4^{k+2}} \) converges.

e) The series \( \sum_{{k=1}}^{\infty} \frac{1}{(1 + k)[\ln(1 + k)]^2} \) converges. (You may assume that the improper integral \( \int_1^{\infty} \frac{dx}{(1 + x)[\ln(1 + x)]^2} \) converges.)

f) \( \left| \sum_{{k=27}}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k}} \right| \leq \frac{1}{3}. \)

2. Find the limits

a) \( \lim_{{x \to 0^-}} (1 - x)^{\frac{2}{3}} \) \hspace{1cm} b) \( \lim_{{x \to 0}} \frac{(1 + x)^{\frac{1}{3}} - 1}{x^2} \)

3. Decide whether the following integrals converge or diverge and evaluate the limits which exist.

a) \( \int_{{0}}^{1} x^{-\frac{4}{3}} \, dx \) \hspace{1cm} b) \( \int_{{0}}^{\infty} \frac{dx}{1 + x^2} \)

4. Determine convergence or divergence for each series below. Indicate clearly the name(s) of the test(s) you are using. Cross out all irrelevant work and circle the work to be graded. **Only the first attempt** that has not been crossed out will be graded.

a) \( \sum_{{k=1}}^{\infty} \frac{k^2 + 9}{2k^4 - k + 3} \) \hspace{1cm} b) \( \sum_{{k=1}}^{\infty} \frac{\cos k}{k^2} \)

c) \( \sum_{{k=1}}^{\infty} \frac{2k}{e^k} \) \hspace{1cm} d) \( \sum_{{k=1}}^{\infty} \frac{\ln k}{k^3} \)
5. For each series below determine whether it converges absolutely, converges conditionally, or diverges.
   a) \( \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k} + 1} \)  
   b) \( \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{2} + \frac{1}{k} \right)^k \)

6. The first swing of the bob of a pendulum is 5 inches. In each subsequent swing the bob travels \( \frac{2}{3} \) of the preceding swing. Find how far the bob will travel before coming to rest.

7. Let \( S = \sum_{1}^{\infty} a_k \) and suppose the partial sums of this series are given by
   \[ S_k = 1 + \frac{\sin 2k}{k}. \]
   What is \( S \)?

8. The first three non-zero terms of the Taylor Series for
   \[ \left( \sum_{k=1}^{\infty} \frac{x^k}{2k} \right) \left( \sum_{k=0}^{\infty} \frac{6}{k!} x^k \right) \]
   are
   a) \( 3 + \frac{9}{2} x + 4x^2 \)  
      b) \( 3x + \frac{9}{2} x^2 + 4x^3 \)  
      c) \( 1 + 3x + \frac{9}{2} x^2 \)
   d) \( \frac{13}{2} x + \frac{51}{4} x^2 + \frac{94}{6} x^3 \)  
      e) none of these

9. The series \( \sum_{k=0}^{\infty} \frac{k^{2k}}{(2k)!} \)
   a) converges by the ratio test  
   b) diverges  
   c) converges by the root test  
   d) converges by the comparison test  
   e) none of these

10. The series \( \sum_{k=20}^{\infty} \frac{1}{\sqrt{4k + 8}} \)
    a) converges by the integral test  
    b) converges by the limit comparison test  
    c) diverges by the limit comparison test  
    d) diverges by the ratio test  
    e) none of these
11. \[ \lim_{x \to \infty} \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^x \]

a) does not exist  
d) equals \( e^2 \)
b) equals \( e \)  
e) none of these
c) equals \( e^{-1} \)

12. \[ \left\{ \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+6} \right\}_{n=1}^{\infty} \]

a) an infinite series which converges  
b) an infinite series which diverges  
c) a sequence which does not converge to 0  
d) a sequence which converges to 0  
e) none of these

13. 

a) Let \( f(x) \) be a function. For the Taylor series of \( f(x) \) about \( a \), write the general formula for \( R_n(x) \), the Lagrange form of the remainder.

b) Find the Taylor series for \( e^{2x} \) about \( a = \ln 3 \). Express your answer in \( \Sigma \) notation.

c) Find \( R_5\left(\frac{1}{2} + \ln 3\right) \) for the particular series in part (b).

14. Let \( f(x) = \sum_{k=2}^{\infty} (-1)^k \frac{(3x)^k}{\ln k} \)

a) Determine the interval of convergence of this series. Show all work.

b) Let \( P_4(x) \) denote the Maclaurin polynomial for \( f \) whose highest power of \( x \) is \( x^4 \). Let \( u \) be the smallest upper bound for \( |f\left(\frac{1}{6}\right) - P_4\left(\frac{1}{6}\right)| \), i.e., number \( u \) such that \( |f\left(\frac{1}{6}\right) - P_4\left(\frac{1}{6}\right)| \leq u \), that you can get by the methods we have studied.

Select the letter from the following list that correctly describes \( u \).
(Answer only will be graded)

a) \( 0 \leq u \leq \frac{1}{36 \ln 5} \)  
b) \( \frac{1}{36 \ln 5} < u \leq \frac{1}{30 \ln 5} \)  
c) \( \frac{1}{30 \ln 5} < u \leq \frac{1}{20 \ln 5} \)

d) \( \frac{1}{20 \ln 5} < u \leq \frac{1}{15 \ln 5} \)  
e) \( u > \frac{1}{15 \ln 5} \)

15. By integrating an appropriate series, find the Maclaurin Series for \( \ln(1 + 4x) \). Express your answer in \( \Sigma \) notation.

16. Calculate the value of \( \cos 58^\circ \) to an accuracy of \( \frac{1}{2 \ln x} \).