1. (14 points) Consider the equation

\[ tx' + (1 - t)x^2 = 0 \]

Please, answer the following questions (NO PARTIAL CREDIT):
(a) Is the given differential equation separable?
ANS: YES
(b) Is it linear?
ANS: NO
(c) Give the largest interval containing \( t = 1 \) where the equation is normal.
ANS: \((0, \infty)\)
(d) Write the equation in the standard form.
\[ x' = -\frac{1-t}{t}x^2 \]
(e) Find the general solution.
Separating variables and integrating, \(-\frac{1}{x} = t - \ln|t| + C\). ANS: \(x = \frac{1}{\ln|t|-t-C}, t \neq 0, \) and \(x = 0\).
(f) Find a solution satisfying the initial condition \(x(1) = -2\).
ANS: \(x = \frac{1}{\ln|t|-t+1/2}\)
(g) Why is the solution satisfying the initial condition \(x(1) = -2\) unique? (Please, refer to an appropriate theorem).
Due to the existence and uniqueness theorem.

2. (10 points) Consider the differential equation

\[ (t - 1) \frac{dx}{dt} = x. \]

(a) Determine the largest rectangular region of the \((t, x)\)-plane that contains the point \((0, 2)\) and on which the hypotheses of the existence and uniqueness theorem hold for the given o.d.e.
ANS: \((-\infty, 1) \times (-\infty, \infty)\)
(b) Determine whether there is no solution, a unique solution, or more than one solutions passing through the point \((1, 2)\).
If there is a solution \(x(t)\) then at \(t = 1\) one has \((1 - 1)x'(1) = x(1), or 0 = 2. This contradiction shows that there is no solution passing through the point \((1, 2)\). ANS: NO SOLUTION.
3 (10 points) Solve the following first order linear differential equation
\[ 2x' - x = te^t. \]

(i) The general solution of the homogeneous equation is \( x = Ce^{\frac{1}{2}t}; \)
(ii) A particular solution to the nonhomogeneous equation is \((t - 2)e^t; \)
(iii) ANS: \( x = Ce^{\frac{1}{2}t} + (t - 2)e^t; \)

4 (8 points) (NO PARTIAL CREDIT)
(a) Determine whether the system of linear algebraic equations
\[
\begin{align*}
  x - 2y + z &= 0 \\
  3x + 2y + z &= 0 \\
  y - z &= 0
\end{align*}
\]
has a unique solution, infinitely many solutions, or no solution;
Cramer’s test implies that the system has a unique solution
(b) If the system has a unique solution, then find this solution.
ANS: (By inspection) \((0, 0, 0)\).

5. (6 points) Use the Wronskian test for independence to show that functions \(e^{at}\) and \(e^{bt}\), with \(a \neq b\), are linearly independent.
The Wronskian is \( W(t) = \det \begin{pmatrix} e^{at} & e^{bt} \\ ae^{at} & be^{bt} \end{pmatrix} = (b - a)e^{(a+b)t} \neq 0 \text{ if } a \neq b. \)

6. (10 points) Use the exponential shift formula to compute the following expressions:
(a) \( (D^2 + D - 5)[te^{2t}] \)
ANS: \( (D^2 + D - 5)[te^{2t}] = e^{2t}(D^2 + D + 5)[t] = e^{2t}(D^2 + 5D + 1)[t] = e^{2t}(5 + t) \).
(b) \( (D - 5)^4[e^{5t} \sin(t)] \)
ANS: \( (D - 5)^4[e^{5t} \sin(t)] = e^{5t}D^4[\sin(t)] = e^{5t} \sin(t) \).

7. (15 points) Consider the following nonhomogeneous second order differential equation
\[ x'' - 2x' + x = t^2. \]

(a) Find the general solution of the corresponding homogeneous equation;
ANS: \( x = (C_1 + tC_2)e^t \)
(b) Find a particular solution of the form \( x(t) = At^2 + Bt + C \) to the nonhomogeneous equation;
ANS: \( x_p = t^2 + 4t + 6 \)
(c) Find the solution of the given equation satisfying the conditions
\[ x(0) = x'(0) = 0. \]
ANS: $x = (2t - 6)e^t + t^2 + 4t + 6$

8. (7 points) Find the annihilator of smallest possible order for the function (NO PARTIAL CREDIT)

$$t^2 + e^t \cos 2t - 1$$

ANS: $A(D) = D^3(D^2 - 2D + 5)$

9. (20 points) Solve the following initial-value problem

$$(D^2 - 1)(D^2 + 1)x = 0$$

$x(0) = 1, x'(0) = x''(0) = x'''(0) = 0$.

The general solution is $x = C_1e^t + C_2e^{-t} + C_3 \cos(t) + C_4 \sin(t)$.

Initial conditions give

$$C_1 + C_2 + C_3 = 1,$$
$$C_1 - C_2 + C_4 = 0,$$
$$C_1 + C_2 - C_3 = 0,$$
$$C_1 - C_2 - C_4 = 0.$$

Adding first and third equations, and then second and forth equations, we have

$$C_1 + C_2 = 1/2,$$
$$C_1 - C_2 = 0.$$

This implies $C_1 = C_2 = 1/4$, and from second and third equations, $C_3 = 1/2$ and $C_4 = 0$.

ANS: $x = \frac{1}{4}e^t + \frac{1}{4}e^{-t} + \frac{1}{2} \cos(t)$. 