Exponential Shift:

\[ P(D) (u e^{xt}) = e^{xt} P(D+1) u \]

5. (d) \( (D^2 + 3D + 2) e^{xt} \)

\[ = e^{xt} \left[ (D-1)^2 + 3(D-1) + 2 \right] e^{xt} \]

\[ = e^{xt} (D^2 + D) e^{xt} \]

\[ = e^{xt} (\lambda + 2\mu) \]

\[ = [\lambda (t+1)] e^{-xt} \]
\[ f(t,x) = -\frac{x}{t-2} \]

\#2

(a) Not continuous at \((2,1)\)

(b) \(f_x(t,x) = -\frac{1}{t-2}\) Not continuous at \((2,1)\)

(c) No guarantee of existence or uniqueness

(d) \[
\frac{dx}{dt} = \frac{x}{t-2} \rightarrow \ln|x| = \ln|t-2| + c
\]
\[
\frac{dx}{x} = \frac{dt}{t-2} \quad x = k(t-2)
\]

No solution satisfying \(x(2) = 1\)

(e) \((2,1)\) is not within a rectangle where \(f\) and \(f_x\) are continuous. Existence is not guaranteed.
#2 (c) \[ \frac{dx}{dt} = -\frac{x}{t-2} \]

\[ f(t,x) = -\frac{x}{t-2} \quad f_x(t,x) = -\frac{1}{t-2} \]

Not continuous at \( t = 2 \).

\( E\subset U \) not satisfied at \( x(2) = 1 \)

(a)

\[ \frac{dx}{x} = -\frac{dt}{t-2}, \quad x \neq 0 \]

\[ \ln|x| = -\ln|t-2| + c \]

\[ x(t) = \frac{k}{t-2} \]

No solution through \((2,1)\)

(e) Rectangles of continuity for \( f \) and \( f_x \) include:

- \( -\infty < t < 2 \)
- \( -\infty < x < \infty \)

and \( 2 < t < \infty \)
- \( -\infty < x < \infty \)

E\&U theorem does not apply at \((2,1)\) because the points are not in either rectangle.
(a) Try $x = t^\alpha$ in

$$(t^2 D^2 - 3t D + x) x = 0$$

$x(x-1)x^\alpha - 3x^\alpha + 2x^\alpha = 0$

$x(x-1)-3x + 2 = 0$

$x^2 - 4x + 2 = 0$

$$x = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$$

$$\alpha = 2 \pm \sqrt{2}$$

(b) Try $x = \alpha t^2 + 2\alpha t + \alpha$ in

$$(D - 1)x = t^\alpha$$

$$(D - 1)\sqrt{x} = \sqrt{t^\alpha}$$

$$\sqrt{\alpha t + 2\alpha - \alpha t^2 - 2\alpha t - 2\alpha} = \pm 2$$

$$-\alpha t^2 = \pm 2$$

$$\alpha = -1$$
#14  Solve $x' - x = e^t \sin t$

(H) $x' - x = 0$

\[
\frac{dx}{x} = dt
\]

$\ln |x| = t + c \quad x = Re^t$

(v) Try $x = R(t)e^t$ in (N)

\[
x' = R^t e^t + R e^t
\]

\[
x' - x = R^t e^t = e^t \sin t
\]

\[
R' = \sin t \quad R(t) = -\cos t + c
\]

\[
x = (c - \cos t)e^t
\]

\[
x(t) = c e^t - e^t \cos t
\]

$\implies c = -c$

2/9/2009
(a) \[
\begin{vmatrix}
\frac{d}{dt} & \sin t & \cos t \\
\cos t & \frac{d}{dt} & -\sin t \\
-\sin t & -\cos t & \frac{d}{dt}
\end{vmatrix} = e^t \begin{vmatrix}
\cos t & -\sin t & 0 \\
-\sin t & -\cos t & 0 \\
0 & 0 & -\sin t & -\cos t
\end{vmatrix}
\]
\[
e^t \left[ (-\cos^2 t + \sin^2 t) - 0 + (-\sin^2 t - \cos^2 t) \right]
\]
\[
= -2e^t
\]

(b) \[
(D^2 + 1) \sin t = 0 \\
(D^2 + 1) \cos t = 0 \\
(D - 1) e^t = 0
\]

$e^t, \sin t, \cos t$ are solutions to the 3rd order, linear ODE \[(H) \quad (D - 1)(D^2 + 1)x = 0\]

By Part (a), the three functions are linearly independent.

$x = c_1 e^t + c_2 \sin t + c_3 \cos t$

is the general solution to (H).
(a) \( (D^3 - 2D^2 + D)x = 0 \) has characteristic polynomial \( p(t) = t^3 - 2t^2 + 1 \)
\[ = t(t^2 - 2t + 1) \]
\[ = t(t-1)^2 \]
with roots \( t = 0, 1, 1 \)

\[
x(t) = c_1 + c_2 e^t + c_3 t e^t
\]

(b) \( x' = (c_2 + c_3) e^t + c_3 t e^t \)
\( x'' = (c_2 + 2c_3) e^t + c_3 t e^t \)
\[ x''(0) = 1 \implies c_2 + 2c_3 = 1 \]
\[ x'(0) = 0 \implies c_2 + c_3 = 0 \]
\[ x(0) = 0 \implies c_1 + c_2 = 0 \]
\[ c_1 = 1, \quad c_2 = -1, \quad c_3 = 1 \]

\[
x(t) = 1 - e^t + t e^t
\]