1. (15 points): Use the method of undetermined coefficients to find a particular solution of the equation
\[ x'' + x' - 2x = -18te^{-2t}. \]

**Solution:** The characteristic equation corresponding to the homogeneous equation is \( \lambda^2 + \lambda - 2 = 0 \). The roots are \( \lambda = 1, \lambda = -2 \), and the general solution to the homogeneous equation is
\[ H(t) = C_1e^t + C_2e^{-2t}. \]

The annihilator for the right side is \( A(D) = (D+2)^2 \). Therefore, any particular solution of the given equation is among the general solution to the equation \( A(D)(D-1)(D+2)x = 0 \), or \( (D - 1)(D + 2)^3x = 0 \). The general solution of this equation is
\[ x = C_1e^t + (C_2 + C_3t + C_4t^2)e^{-2t}. \]

Hence, the simplified form for a particular solution has the form \( x_p = (C_3t + C_4t^2)e^{-2t} \). Substituting this into equation one obtains \( 2C_4 - 3C_3 = 0 \), and \( -6C_4 = -18 \), or \( C_4 = 3, C_3 = 2 \).

**ANS:** \( x_p = (2t + 3t^2)e^{-2t} \).

2. (10 points): Solve the following equation by the method of variation of parameters
\[ 2x'' + 2x = \sec t. \]

No credit by any other method.
Solution: The functions \( h_1(t) = \cos t \) and \( h_2(t) = \sin t \) generate the general solution to the homogeneous equation \( 2x'' + 2x = 0 \). Hence, we look for a particular solution in the form
\[
x_p = c_1(t) \cos t + c_2(t) \sin t.
\]
The system
\[
\begin{align*}
\cos(t)c_1' + \sin(t)c_2' &= 0, \\
-\sin(t)c_1' + \cos(t)c_2' &= \frac{1}{2} \sec t,
\end{align*}
\]
has a solution \( c_1' = -\frac{1}{2} \tan t \) and \( c_2' = \frac{1}{2} \), yielding \( c_1(t) = \frac{1}{2} \ln |\cos t| \) and \( c_2(t) = \frac{t}{2} \).

ANS: \( x(t) = c_1 \cos t + c_2 \sin t + \frac{1}{2} \cos t \ln |\cos t| + \frac{t}{2} \sin t \).

3. (10 points): Find the Laplace transform of the following functions:
   
   (a) \( te^{-t} \sin 2t \)
   
   (b) \[
   \begin{cases} 
   2, & \text{if } t < 2, \\
   t, & \text{if } 2 \leq t < 3, \\
   e^{t-3} + 3, & \text{if } 3 \leq t.
   \end{cases}
   \]

Solution:

(a) By the First Differentiation Formula, \( \mathcal{L}[t \sin(2t)](s) = -\frac{d}{ds} \left[ \frac{2}{s^2 + 4} \right] = \frac{4s}{(s^2 + 4)^2} \).

   Then, by the First Shift Formula, \( \mathcal{L}[e^{-t}t \sin(2t)](s) = \frac{4(s+1)}{((s+1)^2 + 4)^2} \).

(b) \( \mathcal{L}[2 + u_2(t) \cdot (t-2) + u_3(t) \cdot [e^{t-3} + 3 - t]](s) = \frac{2}{s} + e^{-2s} \mathcal{L}[t](s) + e^{-3s} \mathcal{L}[e^t - t](s) = \)
\[
\frac{2}{s} + \frac{e^{-2s}}{s^2} + e^{-3s} \left( \frac{1}{s-1} - \frac{1}{s^2} \right) \text{(Second Shift Formula)}.
\]

4. (10 points): Find the inverse Laplace transform of the following functions:
\[ (a) \quad \frac{e^{-s}(1 + s)}{2(s^2 + 1)} \]
\[ (b) \quad \frac{s}{s^2 - 2s + 2} \]

**Solution:**

(a) \( L^{-1} \left[ \frac{e^{-s}(1 + s)}{2(s^2 + 1)} \right] (t) = \frac{1}{2} u_1(t) (\cos(t - 1) + \sin(t - 1)) \). (Second Shift Formula).

(b) \( L^{-1} \left[ \frac{s}{s^2 - 2s + 2} \right] (t) = L^{-1} \left[ \frac{s - 1}{(s - 1)^2 + 1} + \frac{1}{(s - 1)^2 + 1} \right] (t) = e^t(\cos t + \sin t). \) (First Shift Formula).

5. (10 points): Use the following steps to compute the convolution \( t \ast (\sin t - e^t) \):

(a) find the Laplace transform of \( t \ast (\sin t - e^t) \);

(b) find a partial fraction decomposition of the Laplace transform obtained in Part (a);

(c) find the inverse Laplace transform of the partial fraction decomposition found in Part (b).

**Solution:**

(a) \( \mathcal{L}[t \ast (\sin t - e^t)] = \mathcal{L}[t] \cdot \mathcal{L}[\sin t - e^t] = \frac{1}{s^2} \left( \frac{1}{1 + s^2} - \frac{1}{s-1} \right) \)

(b) \[
\frac{1}{s^2} \left( \frac{1}{1 + s^2} - \frac{1}{s-1} \right) = \frac{1}{s^2(s^2 + 1)} - \frac{1}{s^2(s-1)} = \frac{1}{s^2} - \frac{1}{1 + s^2} - \frac{s-(s-1)}{s^2(s-1)} = \frac{1}{s^2} - \frac{1}{1 + s^2} - \frac{1}{s(s-1)} + \frac{1}{s^3} \\
= \frac{2}{s^3} - \frac{1}{1 + s^2} - \frac{1}{s-1} + \frac{1}{s}.
\]

(c) \( t \ast (\sin t - e^t) = \mathcal{L}^{-1} \left[ \frac{2}{s^3} - \frac{1}{1 + s^2} - \frac{1}{s-1} + \frac{1}{s} \right] = 2t - \sin t - e^t + 1. \) 

**ANS:** \( 2t - \sin t - e^t + 1. \)
6. (10 points): Compute \( \mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right] \) using the convolution formula. No credit by any other method.

**Solution:**

\[
\mathcal{L}^{-1}\left[\frac{1}{s^2(s-2)}\right] (t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] (t) * \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] (t) = t * e^{2t} = \int_0^t u e^{2(t-u)} \, du
\]

\[
= e^{2t} \int_0^t u e^{-2u} \, du = e^{2t}(1 - t/2 - 1/4 e^{-2t}).
\]

**ANS:** \( \frac{e^{2t}}{4} - \frac{t}{2} - \frac{1}{4} \).

---

7. (15 points): Solve the following initial value problem

\[
x'' - 4x' + 5x = f(t),
\]

\[
x(0) = 0, \quad x'(0) = 1,
\]

where \( f(t) = \begin{cases} 
5, & \text{if } t < 1 \\
0, & \text{if } 1 \leq t.
\end{cases} \)
Solution: Applying the Laplace transform to both sides of the equation we have

\[ (s^2 - 4s + 5)\mathcal{L}[x] - 1 = \frac{5}{s} - e^{-s} \frac{5}{s}, \]

so

\[ \mathcal{L}[x] = \frac{1}{(s - 2)^2 + 1} + \frac{5}{s((s - 2)^2 + 1)} - e^{-s} \frac{5}{s((s - 2)^2 + 1)}. \]

Now we have to compute inverse Laplace transforms of these three terms on the right. The first term has the inverse

\[ \mathcal{L}^{-1}\left[ \frac{1}{(s - 2)^2 + 1} \right] = e^{2t} \sin t. \]

Second term can be written in the partial fraction form

\[ \frac{5}{s((s - 2)^2 + 1)} = \frac{1}{s} - \frac{s - 2}{(s - 2)^2 + 1} + \frac{2}{(s - 2)^2 + 1}, \]

therefore,

\[ \mathcal{L}^{-1}\left[ \frac{5}{s((s - 2)^2 + 1)} \right] = 1 - e^{2t} \cos t + 2e^{2t} \sin t. \]

Inverse of the third term due to the Second Shift Formula is

\[ \mathcal{L}^{-1}\left[ e^{-s} \frac{5}{s((s - 2)^2 + 1)} \right] = u_1(t) \cdot [1 - e^{2(t-1)} \cos(t - 1) + 2e^{2(t-1)} \sin(t - 1)]. \]

Hence,

\[ x(t) = 1 - e^{2t} \cos t + 3e^{2t} \sin t - u_1(t) \cdot [1 - e^{2(t-1)} \cos(t - 1) + 2e^{2(t-1)} \sin(t - 1)]. \]

8. (10 points): Reduce the following higher order linear differential equations to equivalent systems (DO NOT SOLVE):

(a) \( x'' + (e^t + t)x' = \sec t \)

(b) \( (D + 1)^3 x = t \)
Solution:
(a) Introducing \( x_1 = x \) and \( x_2 = x' \) we have
\[
\begin{align*}
x_1' &= x_2, \\
x_2' &= -(e^t + t)x_2 + \sec t.
\end{align*}
\]
\textbf{ANS:} \( Dx = Ax + E(t) \) with \( A = \begin{pmatrix} 0 & 1 \\ 0 & -e^t - t \end{pmatrix} \) and \( E(t) = \begin{pmatrix} 0 \\ \sec(t) \end{pmatrix} \)

(b) Introducing \( x_1 = x, \ x_2 = x' \) and \( x_3 = x'' \) we have
\[
\begin{align*}
x_1' &= x_2, \\
x_2' &= x_3, \\
x_3' &= -x_1 - 3x_2 - 3x_3 + t.
\end{align*}
\]
\textbf{ANS:} \( Dx = Ax + E(t) \) with \( A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} \) and \( E(t) = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \).

9. (10 points): Determine whether the vector functions
\[
h_1(t) = \begin{pmatrix} 3e^{4t} \\ e^{4t} \end{pmatrix} \quad \text{and} \quad h_2(t) = \begin{pmatrix} 3e^{-4t} \\ e^{-4t} \end{pmatrix}
\]
generate the general solution of the system
\[
\begin{align*}
x_1' &= 5x_1 - 3x_2, \\
x_2' &= 3x_1 - 5x_2.
\end{align*}
\]
Justify your answer.
Solution: (a) We can check that $h_2(t)$ does not satisfy the system,
(b) The Wronskian of these two vector functions is 0 for all $t$.
Either of these imply that the given vector functions do not generate the general solution.