MR2371607 37Fxx (37Bxx 37Exx)

Hasselblatt, Boris (1-TUFT); **Propp, James** (1-MAL)

Corrigendum to: "Degree-growth of monomial maps" [Ergodic Theory Dynam. Systems 27 (2007), no. 5, 1375–1397; MR2358970]. (English summary)

Ergodic Theory Dynam. Systems **27** (2007), *no. 6,* 1999.

{A review for this item is in process.}

MR2358970 37Fxx (37Bxx 37Exx)

Hasselblatt, Boris (1-TUFT); **Propp, James** (1-MAL)

Degree-growth of monomial maps. (English summary)

Ergodic Theory Dynam. Systems **27** (2007), *no. 5,* 1375–1397.

{A review for this item is in process.}

References

1. N. Abarenkova, J.-C. Anglès d'Auriac, S. Bourkraa and J.-M. Maillard. Growth-complexity spectrum of some discrete dynamical systems. *Phys.* D **130** (1999), 27–42. MR1694727 (2000m:37193)
2. R. L. Adler, A. G. Konheim and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.* **114** (1965), 309–319. MR0175106 (30 #5291)
3. V. I. Arnold. Dynamics of complexity of intersections. *Bol. Soc. Brasil. Mat.* **21** (1990), 1–10. MR1139553 (93c:58031)
4. E. Bedford and K. Kim. On the degree growth of birational mappings in higher dimension. *J. Geom. Anal.* **14** (2004), 567–596. MR2111418 (2006b:14018)
5. M. P. Bellon. Algebraic entropy of birational maps with invariant curves. *Lett. Math. Phys.* **50** (1999), 79–90. MR1751620 (2001b:37066)
6. M. P. Bellon and C.-M. Viallet. Algebraic entropy. *Comm. Math. Phys.* **204**(2) (1999), 425–437 (chao-dyn/9805006). MR1704282 (2000f:37040)
7. S. Boucksom, C. Favre and M. Jonsson. Degree growth of meromorphic surface maps. *Duke Math. J.* to appear (math.DS/0608267).
8. S. Bourkraa and J.-M. Maillard. Factorization properties of birational mappings. *Phys.* A **220**

(1995), 403–470.
9. R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** (1971), 401–414; Erratum, **181** (1973), 509–510. MR0274707 (43 #469)
10. J. Diller and C. Favre. Dynamics of bimeromorphic maps of surfaces. *Amer. J. Math.* **123** (2001), 1135–1169. MR1867314 (2002k:32028)
11. E. I. Dinaburg. The relation between topological entropy and metric entropy. *Sov. Math. Dokl.* **11** (1970), 13–16.
12. T.-C. Dinh and N. Sibony. Une borne supérieure pour l'entropie topologique d'une application rationnelle. *Ann. of Math.* (2) **161** (2005), 1637–1644. MR2180409 (2006f:32026)
13. G. Falqui and C.-M. Viallet. Singularity, complexity, and quasi-integrability of rational mappings. *Comm. Math. Phys.* **154** (1993), 111–125. MR1220950 (94i:14019)
14. C. Favre. Les applications monomiales en deux dimensions. *Michigan Math. J.* **51** (2003), 467–475. MR2021001 (2004i:32033)
15. C. Favre and M. Jonsson. Eigenvaluations. *Preprint*, 2004, math.DS/0410417. cf. MR2339287
16. S. Fomin and N. Reading. Root systems and generalized associahedra. *Preprint*, 2005, math.CO/0505518. cf. MR 2006g:05230
17. S. Fomin and A. Zelevinsky. The Laurent phenomenon. *Adv. Appl. Math.* **28**(2) (2002), 119–144 (math.CO/0104241). MR1888840 (2002m:05013)
18. S. Friedland and J. Milnor. Dynamical properties of plane polynomial automorphisms. *Ergod. Th. & Dynam. Sys.* **9** (1989), 67–99. MR0991490 (90f:58163)
19. S. Friedland. Entropy of rational selfmaps of projective varieties. *Proc. Int. Conf. on Dynamical Systems and Related Topics (Advanced Series on Dynamical Systems, 9).* Ed. K. Shiraiwa. World Scientific, Singapore, 1990, pp. 128–140. MR1171886 (94e:32048)
20. S. Friedland. Entropy of polynomial and rational maps. *Ann. of Math.* (2) **133** (1991), 359–368. MR1097242 (92c:58115)
21. S. Friedland. Entropy of algebraic maps. *J. Fourier Anal. Appl. (Kahane Special Issue)* (1995), 215–228. MR1364887 (97k:28040)
22. S. Friedland. Entropy of holomorphic and rational maps: a survey. *Dynamics, Ergodic Theory and Geometry.* Cambridge University Press, Cambridge, 2006.
23. D. Gale. The strange and surprising saga of the Somos sequences. *Math. Intelligencer* **13** (1991), 40–42 and Somos Sequence Update, *Math. Intelligencer* **13,** (1991), 49–50. (Republished in D. Gale. *Tracking the Automatic Ant.* Springer, Berlin, 1998, pp. 2–5, 22–24.)
24. B. Grammaticos, A. Ramani and V. G. Papageorgiou. Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.* **67** (1991), 1825–1826. MR1125950 (92f:58081)
25. V. Guedj. Entropie topologique des applications méromorphes. *Ergod. Th. & Dynam. Sys.* **25**(6) (2005), 1847–1855. MR2183297 (2006f:37064)
26. R. Hartshorne. *Algebraic Geometry.* Springer, Berlin, 1977. MR0463157 (57 #3116)
27. B. Hasselblatt, Z. Nitecki and J. Propp. Topological entropy for nonuniformly continuous maps. *Discrete Contin. Dyn. Syst.* (Yakov Pesin Birthday Issue) to appear (math.DS/0511495).
28. J. Hietarinta and C.-M. Viallet. Singularity confinement and chaos in discrete systems. *Phys. Rev. Lett.* **81** (1998), 325–328.
29. J. Hientarinta and C. Viallet. Singularity confinement and degree growth. *Symmetries and*

Integrability of Difference Equations (SIDE III) (CRM Proceedings and Lecture Notes, 25). Eds. D. Levi and O. Ragnisco. American Mathematical Society, Providence, RI, 2000, pp. 209–216. Available from http://www.lpthe.jussieu.fr/∼viallet/. MR1771723 (2001m:39052)

30. A. Hone. Diophantine nonintegrability of a third order recurrence with the Laurent property. *J. Phys. A: Math. Gen.* **39** (2006), L171—L177. MR2212057 (2006j:11017)
31. A. Hone. Singularity confinement for maps with the Laurent property. *Phys. Lett.* A **361** (2007), 341–345 (nlin.SI/0602007).
32. A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995. MR1326374 (96c:58055)
33. S. Lafortune, A. Ramani, B. Grammaticos, Y. Ohta and K. M. Tamizhmani. Blending two discrete integrability criteria: singularity confinement and algebraic entropy. *Bäcklund and Darboux Transformations. The Geometry of Solitons (Halifax, NS, 1999) (CRM Proc. Lecture Notes, 29)*. American Mathematical Society, Providence, RI, 2001, pp. 299–311 (nlin.SI/0104020). MR1870928 (2003e:39030)
34. D. Lind and T. Ward. Automorphisms of solenoids and p-adic entropy of dynamical systems. *Ergod. Th. & Dynam. Sys.* **8** (1988), 411–419. MR0961739 (90a:28031)
35. K. Maegawa. Quadratic polynomial automorphisms of dynamical degree golden ratio of C^3. *Ergod. Th. & Dynam. Sys.* **21** (2001), 823–832. MR1836434 (2002e:32026)
36. D. Mumford. *Algebraic Geometry, Vol. 1: Complex Projective Varieties*. Springer, Berlin, 1995. MR1344216 (96d:14001)
37. G. Musiker and J. Propp. Combinatorial interpretations for rank-two cluster algebras of affine type. *Electron. J. Combin.* **14** (2007), paper 15 (math.CO/0602408). MR2285819
38. V. Nguyén. Algebraic degrees for iterates of meromorphic self-maps of $\mathbb{P}^k$. *Publ. Mat.* **50**(2) (2006), 457–473 (math.CV/0603545). MR2273670
39. D. V. Osin. Algebraic entropy of elementary amenable groups. *Geom. Dedicata* **107** (2004), 133–151 (math.GR/0404075). MR2110759 (2006b:37010)
40. A. Ramani, B. Grammaticos, S. Lafortune and Y. Ohta. Linearisable mappings and the low-growth criterion. *J. Phys. A: Math. Gen.* **33**(31) (2000), L287—L292 (nlin.SI/0104015). MR1784222 (2001h:37038)
41. A. Russakovskii and B. Shiffman. Value distribution for sequences of rational mappings and complex dynamics. *Indiana Univ. Math. J.* **46** (1997), 897–932. MR1488341 (98h:32046)
42. P. Sherman and A. Zelevinsky. Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. *Moscow Math. J.* **4** (2004), 947–974. MR2124174 (2006c:16052)
43. D. Speyer and B. Sturmfels. Tropical mathematics. *Preprint*, 2004, math.CO/0408099 (Notes prepared for *Clay Mathematics Institute Senior Scholar Lecture*, by Bernd Sturmfels, Park City, UT, 22 July 2004). cf. MR 2005d:14089
44. R. P. Stanley. *Enumerative Combinatorics* Vol. 1. With a foreword by Gian-Carlo Rota. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986. MR0847717 (87j:05003)
45. T. Takenawa. A geometric approach to singularity confinement and algebraic entropy. *J. Phys. A: Math. Gen.* **34**(10) (2001), L95—L102 (nlin.SI/0011037). MR1828835 (2002c:14029)
46. T. Takenawa. Algebraic entropy and the space of initial values for discrete dynamical systems.

Symmetries and Integrability of Difference Equations (Tokyo, 2000). J. Phys. A: Math. Gen. **34**(48) (2001), 10 533–10 545 (nlin.SI/0103011). MR1877473 (2002m:32026)

47. T. Takenawa, M. Eguchi, B. Grammaticos, Y. Ohta, A. Ramani and J. Satsuma. The space of initial conditions for linearisable mappings. *Nonlinearity* **16**(2) (2003), 457–477 (nlin.SI/02040070). MR1958611 (2004a:37055)
48. A. P. Veselov. Integrable maps. *Russian Math. Surveys* **46**(5) (1991), 1–51. Russian original: *Uspekhi Mat. Nauk.* **46**(5) (1991), 3–45. MR1160332 (93e:58096)
49. H. Wielandt. Unzerlegbare, nicht negative Matrizen (Engl.: Irreducible non-negative matrices). *Math. Z.* **52** (1950), 642–648. MR0035265 (11,710g)
50. A. Zelevinsky. Semicanonical basis generators of the cluster algebra of type $A_1^{(1)}$. *Preprint*, 2006, math.RA/0606775. cf. MR2285807

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2226480 37-06 (01A70)
Hasselblatt, Boris
Preface [Dedicated to Anatole Katok on the occasion of his 60th birthday].
Discrete Contin. Dyn. Syst. **16** (2006), *no. 2, i*.

{There will be no review of this item.}

MR2186241 (2006k:37070) 37D30 (37C40)
Hasselblatt, Boris (1-TUFT); **Pesin, Yakov** (1-PAS)
Partially hyperbolic dynamical systems.
Handbook of dynamical systems. Vol. 1*B*, 1–55, *Elsevier B. V., Amsterdam*, 2006.

In this survey the authors discuss several topics in the theory of partially hyperbolic systems including:
— definition of partial hyperbolicity (PH) (including the broad sense);
— Hölder continuity of invariant distributions of PH-systems;

— examples including frame flows, direct and skew products, group extensions;
— the Brin-Burago-Ivanov theorem concerning the infinite cardinality of the fundamental group of compact 3-manifolds supporting PH diffeomorphisms;
— Pujals' list of PH systems on 3-manifolds;
— the Mather spectrum and Katok's example of non-absolute continuity of central foliations (a central problem to apply the Hopf argument to prove ergodicity);
— stable and unstable foliations and their filtrations;
— the integrability problem for central foliations;
— dynamical coherence;
— robustness and plaque expansiveness of central foliations;
— accessibility and transitivity (There is a nice exposition on p. 35 explaining why accessibility is necessary to obtain ergodicity of PH systems via the Hopf argument.);
— the Dolgopyat-Wilkinson Theorem concerning the C^1 genericity of stably accessible systems;
— stable accessibility from joint integrability for time-1 maps;
— the authors' conjecture concerning stable accessibility of PH systems with the accessibility property;
— the authors' conjecture concerning the C^r-open denseness of stable accessible PH systems;
— the ergodicity of volume-preserving essentially accessible PH systems with one-dimensional central subbundle;
— Hertz's result proving stable ergodicity for all linear automorphisms on the n-torus, $n \geq 5$;
— SRB measures for PH attractors.

Unfortunately the authors do not consider other group actions, such as the flow case.

{For the entire collection see MR2184980 (2006f:37003)}

Reviewed by *Carlos A. Morales*

MR2184980 (2006f:37003) 37-06 (28Dxx 35-06 37-02 37Axx 37C40 37Dxx)

★**Handbook of dynamical systems. Vol. 1B.**

Edited by B. Hasselblatt and A. Katok.

Elsevier B. V., Amsterdam, 2006. *xii*+1222 *pp*. $215.00. *ISBN* 0-444-52055-4

{Vol. 1A has been reviewed [MR1928517 (2003c:37002)].}

Contents: Boris Hasselblatt and Yakov Pesin, Partially hyperbolic dynamical systems (1–55) MR2186241 (2006k:37070); Luis Barreira and Yakov Pesin, Smooth ergodic theory and nonuniformly hyperbolic dynamics (57–263) MR2186242 (2007c:37023); Stefano Luzzatto, Stochastic-like behaviour in nonuniformly expanding maps (265–326) MR2186243 (2007c:37024); Enrique R. Pujals and Martin Sambarino, Homoclinic bifurcations, dominated splitting, and robust transitivity (327–378) MR2186244 (2007d:37042); Yuri Kifer and Pei-Dong Liu, Random dynamics (379–499) MR2186245 (2008a:37002); Pascal Hubert and Thomas A. Schmidt, An introduction to Veech surfaces (501–526) MR2186246; Howard Masur, Ergodic theory of translation surfaces (527–547) MR2186247; Giovanni Forni, On the Lyapunov exponents of the Kontsevich-Zorich cocycle (549–580) MR2186248; Alex Eskin, Counting problems in moduli space (581–595) MR2186249 (2007e:37001); E. Glasner and B. Weiss [Benjamin Weiss], On the interplay between measurable and topological dynamics (597–648) MR2186250; Anatole Katok and Jean-Paul Thouvenot, Spectral properties and combinatorial constructions in ergodic theory (649–743) MR2186251 (2006k:37002); Vitaly Bergelson, Combinatorial and Diophantine applications of ergodic theory (745–869) MR2186252 (2006j:37006); Amos Nevo, Pointwise ergodic theorems for actions of groups (871–982) MR2186253 (2006k:37006); A. V. Babin, Global attractors in PDE (983–1085) MR2186254 (2006h:37107); Sergei B. Kuksin, Hamiltonian PDEs (1087–1133) MR2186255 (2006k:37170); M. I. Weinstein, Extended Hamiltonian systems (1135–1153) MR2186256 (2007b:35049).

{The papers are being reviewed individually.}

MR2124175 01A70

Brin, M. [Brin, Michael]; **Hasselblat, B. [Hasselblatt, Boris]**; **Ilyashenko, Yu.**; **Kushnirenko, A. [Kushnirenko, A. G.]**; **Pesin, Ya.**; **Sossinski, A. [Sosinskiĭ, A. B.]**; **Tsfasman, M.**

Anatole Katok.

Mosc. Math. J. **4** (2004), *no. 4,* 977–979.

{There will be no review of this item.}

MR2093308 (2007j:37037) 37C45 (37D05)
Hasselblatt, Boris (1-TUFT); **Schmeling, Jörg** (S-LUNDT)
Dimension product structure of hyperbolic sets. (English summary)
Modern dynamical systems and applications, 331–345, *Cambridge Univ. Press, Cambridge*, 2004.

In this article the authors consider the problem of computing the dimension of hyperbolic sets in terms of their stable and unstable slices. It is conjectured that the dimension (Hausdorff or upper box) of a hyperbolic set is the sum of the dimensions of the stable and unstable slices. The authors propose an approach towards establishing this conjecture which consists in proving that all the stable and unstable slices have the same dimension. This fact is essential for making the statement of the conjecture meaningful. The results are established for the specific case of the Smale solenoid; namely, the underlying manifold is $M=\mathbb{S}^1\times\mathbb{D}$, where $\mathbb{D}=\{v\colon |v|<1\}$ and the dynamics given by the C^2 map $f\colon M\to M$ is defined as

$$f(x,y,z)=(\eta x,\lambda y+u(x),\mu z+v(x)),$$

with $\eta,\lambda,\mu\colon M\to\mathbb{R}^+$ satisfying $\mu<\lambda<1<\eta$. The stable slices are defined using the natural projection $\pi_x\colon(x,y,z)\mapsto x$, as follows. If one considers the attractor $\Lambda=\bigcap_{n\in\mathbb{N}}f^n(M)$, then the stable manifold in p is $W^{\rm s}(p)=\Lambda_{\pi_x(p)}$, where for a set $A\subset M$ one puts $A_x\colon=(\pi_x|_A)^{-1}(x)$. The stable slices are $\Lambda_x=(\pi_x|_\Lambda)^{-1}(x)$. If λ, μ and η are assumed to be constant, and, in particular, $\eta=2$, the authors first prove that

(i) $\dim\Lambda_x=\dim\Lambda_y$ for any $x,y\in\mathbb{S}^1$, and
(ii) $\dim\Lambda=\dim\Lambda_x+1$.

This result (Theorem 2) is established under the hypothesis of transversality of the unstable foliation. However, in Theorem 5 it is shown that, when f is analytic, the dimension of the slices is independent of the slice, stable or unstable, without requiring transversality and when f is analytic. The proof is deduced from results about the existence of a bi-Lipschitz correspondence between any two slices, which is obtained from holonomy maps. The set of bi-Lipschitz points for the holonomy map between local stable manifolds has full dimension and so preserves the transverse dimension. Theorem 2 follows as a consequence of this result. Using this theorem as a tool, the authors study the set of points p in which the holonomy from $W^{\rm s}(p)$ to another manifold $W^{\rm s}(q)$ is not Lipschitz. It is proved that the entropy of this set is less than that of the attractor Λ.

{For the entire collection see MR2090761 (2005f:37001)}

Reviewed by *Alejandro Mario Mesón*

MR2090762 01A70

Pesin, Yakov (1-PAS); **Brin, Michael** (1-MD); **Hasselblatt, Boris** (1-TUFT)

Anatole Katok.

Modern dynamical systems and applications, xi–xiv, *Cambridge Univ. Press, Cambridge*, 2004.

{This item will not be reviewed individually. For details of the collection in which this item appears see MR2090761 (2005f:37001) .}

{For the entire collection see MR2090761 (2005f:37001)}

MR2090761 (2005f:37001) 37-06 (00B30 22-06)

★**Modern dynamical systems and applications.**

Dedicated to Anatole Katok on his 60th birthday.

Edited by Michael Brin, Boris Hasselblatt and Yakov Pesin.

Cambridge University Press, Cambridge, 2004. *x*+458 *pp*. $90.00. *ISBN* 0-521-84073-2

Contents: Michael Brin, Boris Hasselblatt and Yakov Pesin, Anatole Katok (xi–xiv) MR2090762; V. Bergelson and A. Gorodnik, Weakly mixing group actions: a brief survey and an example (3–25) MR2090763 (2005m:37010); Mélanie Bertelson and Misha Gromov, Dynamical Morse entropy (27–44) MR2090764 (2005m:37056); M. Boyle [Mike Boyle] and J. B. Wagoner, Positive algebraic K-theory and shifts of finite type (45–66) MR2090765 (2005j:37016); Patrick Eberlein, Geometry of 2-step nilpotent Lie groups (67–101) MR2090766 (2005m:53081); R. Feres, A differential-geometric view of normal forms of contractions (103–121) MR2090767 (2005g:37051); Bernard Host and Bryna Kra, Averaging along cubes (123–144) MR2090768 (2005h:37004); Chao-Hui Lin and Daniel Rudolph, Sections for semiflows and Kakutani shift equivalence (145–161) MR2090769 (2006f:37004); Herbert Abels and Gregory Margulis [Grigorii A. Margulis], Coarsely geodesic metrics on reductive groups (163–183) MR2090770 (2006e:22011); Klaus Schmidt [Klaus Schmidt[1]], Algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups (185–209) MR2090771 (2005g:37003); R. J. Spatzier, An invitation to rigidity theory (211–231) MR2090772 (2006a:53041); Serge Tabachnikov, Remarks on magnetic flows and magnetic billiards, Finsler metrics and a magnetic analog of Hilbert's fourth problem (233–250) MR2090773 (2005k:37073); Alexander Blokh, Chris Cleveland and Michał Misiurewicz, Expanding polymodials (253–270) MR2090774 (2006d:37076); Jairo Bochi and Marcelo Viana, Lyapunov exponents: how frequently are dynamical systems hyperbolic? (271–297) MR2090775 (2005g:37060); Christian Bonatti and John Franks, A Hölder continuous vector field tangent to many foliations (299–306) MR2090776 (2005h:37057); M. Brin [Michael Brin], D. Burago and S. Ivanov [Sergei Vladimirovich Ivanov], On partially hyperbolic diffeomorphisms of 3-manifolds with commutative fundamental group (307–312) MR2090777 (2005g:37064); Dmitry Dolgopyat, Prelude to a kiss (313–324) MR2090778 (2005h:37087); Bassam Fayad, Nonuniform hyperbolicity and elliptic dynamics (325–330) MR2093307 (2005m:37065); Boris Hasselblatt and Jörg Schmeling, Dimension product structure of hyperbolic sets (331–345) MR2093308 (2007j:37037); Huyi Hu, Yakov Pesin and Anna Talitskaya, Every compact manifold carries a hyperbolic Bernoulli flow (347–358) MR2093309 (2005g:37062); Michael Jakobson, Parameter choice for families of maps with many critical points (359–364) MR2093310 (2005j:37050); Oliver Jenkinson and Mark Pollicott, Entropy, exponents and invariant densities for hyperbolic systems: dependence and computation (365–384) MR2093311 (2005h:37066); Yuri Kifer, Some recent advances in averaging (385–403) MR2093312 (2005h:37067); R. de la Llave, Bootstrap of regularity for integrable solutions of cohomology equations (405–418) MR2093313 (2005g:37063); Sheldon Newhouse, Cone-fields, domination, and hyperbolicity (419–432) MR2093314 (2005i:37034); Domokos Szász and Tamás Varjú, Markov towers and stochastic properties of billiards (433–445) MR2093315 (2005g:37074); Jean-Christophe Yoccoz, Some questions and remarks about $\mathrm{SL}(2,\mathbf{R})$ cocycles (447–458) MR2093316 (2005h:37068).

{Most of the papers are being reviewed individually.}

MR2084468 (2006e:37034) 37C45 (37D05 37D10)

Hasselblatt, Boris (1-TUFT); **Schmeling, Jorg** (S-LUND-LIT)

Dimension product structure of hyperbolic sets. (English summary)

Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 88–96 (*electronic*).

In this announcement the authors conjecture that the fractal dimension of a hyperbolic set is (at least generically or under mild hypotheses) the sum of those of its stable and unstable slices, where “fractal” can mean either Hausdorff or upper box dimension. This would facilitate substantial progress in the calculation or estimation of these dimensions, which are related in deep ways to dynamical properties. The authors prove the conjecture in a model case of Smale solenoids.

For hyperbolic invariant measures a somewhat similar conjecture (the Eckmann-Ruelle conjecture) was proved in [L. M. Barreira, Y. B. Pesin and J. Schmeling, Ann. of Math. (2) **149** (1999), no. 3, 755–783; MR1709302 (2000f:37027)] by establishing a dimension product structure. For the case of Hausdorff or upper box dimension, this kind of structure is difficult to establish since the homology maps along stable or unstable lamination may not be Lipschitz continuous. The authors describe possible ways to overcome this difficulty in proving the conjecture for the model case of Smale solenoids.

Reviewed by *Pei Dong Liu*

References

1. DmitriĭAnosov: *Geodesic flows on Riemann manifolds with negative curvature*, Proc. Steklov Inst. **90** (1967). MR 0224110 (36:7157)
2. DmitriĭAnosov, Yakov Sinai: *Some smooth ergodic systems*, Russian Math. Surveys **22** (1967), no. 5, 103–167. MR 0224771 (37:370)
3. L. Barreira and J. Schmeling: *Sets of "non-typical" points have full topological en- tropy and full Hausdorff dimension*, Israel Journal of Mathematics **116** (2000), 29–70. MR1759398 (2002d:37040)
4. Lu'ıs Barreira, Yakov Pesin, J"org Schmeling: *On the pointwise dimension of hyperbolic measures: a proof of the Eckmann-Ruelle conjecture*, Electronic Research Announce- ments of the AMS **2** (1996), no. 1, 69–72 (electronic); *Dimension and product struc- ture of hyperbolic measures*, Annals of Mathematics (2) **149** (1999), no. 3, 755–783. MR1405971 (97i:58094)
5. Hans-G"unter Bothe: *The dimension of some solenoids*, Ergodic Theory and Dynamical Systems **15** (1995), no. 3, 449–474 MR1336701 (96h:28005)
6. Rufus Bowen, *Topological entropy for noncompact sets*, Transactions of the AMS **184** (1973), 125–136. MR0338317 (49 #3082)
7. Michael Brin, Garrett Stuck: *Introduction to dynamical systems*. Cambridge University Press, Cambridge, 2002. MR1963683 (2003m:37001)
8. Kenneth Falconer: *The geometry of fractal sets*. Cambridge University Press, Cambridge, 1986. MR0867284 (88d:28001)
9. Boris Hasselblatt: *Regularity of the Anosov splitting and of horospheric foliations*, Ergodic Theory and Dynamical Systems **14** (1994), no. 4, 645–666. MR1304137 (95j:58130)
10. Boris Hasselblatt: *Hyperbolic dynamics*, in *Handbook of Dynamical Systems 1A*, Elsevier, 2002. MR 1928517 (2003c:37002)
11. Boris Hasselblatt, J"org Schmeling: *Dimension product structure of hyperbolic sets*. In *Modern Dynamical Systems and Applications*, B. Hasselblatt, M. Brin, Y. Pesin, eds., Cambridge University Press, New York, 2004, pp. 331–345. MR2093308 (2007j:37037)
12. Boris Hasselblatt, Amie Wilkinson: *Prevalence of non-Lipschitz Anosov foliations*, Er- godic Theory and Dynamical Systems **19** (1999), no. 3, 643–656; ERA-AMS 3 (1997), 93–98. MR1695913 (2000f:37035)
13. Eberhard Hopf: *Statistik der geod"atischen Linien in Mannigfaltigkeiten negativer Kr"ummung*, Berichte "uber die Verhandlungen der S"achsischen Akademie der Wis- senschaften zu Leipzig, Mathematisch-Physikalische Klasse **91** (1939), 261–304. MR0001464 (1,243a)
14. Anatole Katok, Boris Hasselblatt: *Introduction to the modern theory of dynamical sys- tems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, 1995. MR1326374 (96c:58055)
15. Fran¸cois Ledrappier, Lai-Sang Young: *The metric entropy of diffeomorphisms. I. Charac- terization of measures satisfying Pesin's entropy formula; II. Relations between entropy, exponents and dimension*, Annals of Mathematics (2) **122** (1985), no. 3, 509–539; 540–574. MR0819556 (87i:58101a)
16. Ya. Pesin, *Dimension theory in dynamical systems: contemporary views and applications*,

Chicago Lectures in Mathematics, Chicago University Press, 1997. MR1489237 (99b:58003)

17. Charles C. Pugh, Michael Shub, Amie Wilkinson: *H"older foliations*, Duke Math. J. **86** (1997), no. 3, 517–546. MR1432307 (97m:58155)
18. J"org Schmeling: *H"older continuity of the holonomy maps for hyperbolic basic sets. II*, Mathematische Nachrichten **170** (1994), 211–225. MR1302376 (95m:58098)
19. J"org Schmeling, Rainer Siegmund-Schultze: *H"older continuity of the holonomy maps for hyperbolic basic sets. I.*, Ergodic theory and related topics, III (G"ustrow, 1990), pp. 174–191, Springer Lecture Notes in Mathematics **1514**, Springer-Verlag, Berlin, 1992. MR1179182 (93j:58104)
20. Steven Smale: *Differentiable dynamical systems*, Bulletin of the AMS **73** (1967), 747–817. MR0228014 (37 #3598)
21. Lai-Sang Young: *What are SRB measures, and which dynamical systems have them?*, J. Statistical Physics **108** (2002), no. 5/6, 733–754. MR1933431 (2003g:37042)

Note: The reference list for this item has been removed for technical reasons.

MR2031955 (2005b:37049) 37D30 (37C10 37D10 37E99)
Foulon, Patrick (F-STRAS-I); **Hasselblatt, Boris** (1-TUFT)
Zygmund strong foliations. (English summary)
Israel J. Math. **138** (2003), 157–169.

The authors consider volume-preserving Anosov flows (which are C^k with $k \geq 2$) on three-dimensional manifolds. S. E. Hurder and A. B. Katok [Inst. Hautes Études Sci. Publ. Math. No. 72 (1990), 5–61 (1991); MR1087392 (92b:58179)] showed that the weak stable and weak unstable foliations are C^1 with a derivative in the Zygmund class. They also studied the obtruction to higher differentiability and related it to a cocycle.

In this paper, the authors prove that the strong stable and strong unstable bundles E^s, E^u are in the Zygmund class. They also exhibit a cocycle $K\colon M \to \mathbb{R}$ which gives an obstruction to higher differentiability: the cocycle K is a coboundary if and only if the subbundle $E^u \oplus E^s$ is little-Zygmund (which is stronger than Zygmund regularity but weaker than beeing C^1). In this case, $E^u \oplus E^s$ is in fact C^{k-1} and the flow is a suspension or a contact flow.

Reviewed by *Sylvain Crovisier*

References

1. D. DeLatte, *Nonstationary normal forms and cocycle invariants*, Random and Computational Dynamics **1** (1992/93), 229–259; *On normal forms in Hamiltonian dynamics, a new approach*

to some convergence questions, Ergodic Theory and Dynamical Systems **15** (1995), 49–66. MR1186375 (93m:58102)

2. J. S. Hadamard, *Sur l'itération et les solutions asymptotiques des équations différentielles*, Bulletin de la Société Mathématique de France **29** (1901), 224–228.
3. B. Hasselblatt, *Regularity of the Anosov splitting and of horospheric foliations*, Ergodic Theory and Dynamical Systems **14** (1994), 645–666. MR1304137 (95j:58130)
4. S. Hurder and A. Katok, *Differentiability, rigidity, and Godbillon–Vey classes for Anosov flows*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **72** (1990), 5–61. MR1087392 (92b:58179)
5. A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, 1995. MR1326374 (96c:58055)
6. A. Livšic, *Certain properties of the homology of Y-systems*, Akademiya Nauk SSSR. Matematicheskie Zametki **10** (1971), 555–564; *Cohomology of dynamical systems*, Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya **36** (1972) 1296–1320. MR0293669 (45 #2746)
7. W. F. Pfeffer, *Derivation and integration*, Cambridge Tracts in Mathematics, Cambridge University Press, 2001. MR1816996 (2001m:26018)
8. J. F. Plante, *Anosov flows*, American Journal of Mathematics **94** (1972), 729–754. MR0377930 (51 #14099)
9. A. S. Zygmund, *Trigonometric Series*, Cambridge University Press, 1959 (and 1968, 1979, 1988), revised version of *Trigonometrical Series*, Monografje Matematyczne, Tom V, Warszawa-Lwow, 1935. MR0933759 (89c:42001)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2016702 (2004m:34098) 34D05 (34H05 37C10 37D20)

Gerber, Marlies (1-IN); **Hasselblatt, Boris** (1-TUFT); **Keesing, Daniel** (1-TUFT)

The Riccati equation: pinching of forcing and solutions. (English summary)

Experiment. Math. **12** (2003), *no. 2,* 129–134.

This paper deals with solutions of the Riccati equation $\dot{x} + x^2 = k(t)$ where x and k are real-valued functions with $k(t) > 0$. The main thrust of the article is the comparison of solutions for different functions k and a discussion of the best possible nature of the results. The new results of the paper are too long to give here, but are related to the following result of B. Hasselblatt [J. Differential Geom. **39** (1994), no. 1, 57–63; MR1258914 (95c:58137)]: If $a \in (0, 1)$, $0 < ak_1(t) <$

$k_2(t) < k_1(t)$ and x_i is a solution of $\dot{x} + x^2 - k_i(t)$ for $i = 1, 2$, then $x_2(0) > ax_1(0) > 0$ implies that $x_2(t) > ax_1(t)$ for all $t > 0$.

Reviewed by *Bernard J. Harris*

References

1. Dmitriĭ Viktorovich Anosov and Yakov Sinai. ”Some Smooth Ergodic Systems.” *Russian Mathematical Surveys* 22: 5 (1967), 103–167.
2. Richard Bellman. *Stability Theory of Differential Equations*. New York-Toronto-London: McGraw-Hill, 1953. MR0061235 (15,794b)
3. Sergio Bittanti, Alan J. Laub, and Jan C. Willems. *The Riccati equation*. Berlin: Springer-Verlag, 1991 MR1132048 (92d:93008)
4. Harold T. Davis. *Introduction to Nonlinear Differential and Integral Equations*. New York: Dover, 1990. MR0181773 (31 #6000)
5. Neil Fenichel. ”Persistence and Smoothness of Invariant Manifolds for Flow.” *Indiana University Mathematics Journal* 21 (1971/1972), 193–226; ”Asymptotic Stability with Rate Conditions.” *Indiana University Mathematics Journal* 23 (1974), 1109–1137; 26:1 (1977), 81–93; ”Hadamard’s Generalization of Hyperbolicity, with Applications to the Hopf Bifurcation Problem.” In *Dynamical Systems, Theory and Applications* (Rencontres, Battelle Res. Inst., Seattle, WA, 1974), pp. 525–538, Lecture Notes in Physics 38. Berlin-New York: Springer-Verlag 1975. MR0455044 (56 #13285)
6. Jaap L. Geluk and Laurens de Haan. *Regular Variation, Extensions and Tauberian Theorems*, CWI Tract 40. Amsterdam: Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, 1987. MR0906871 (89a:26002)
7. Boris Hasselblatt. ”Regularity of the Anosov Splitting and of Horospheric Foliations.” *Ergodic Theory and Dynamical Systems* 14:4 (1994), 645–666. MR1304137 (95j:58130)
8. Boris Hasselblatt. ”Horospheric Foliations and Relative Pinching.” *Journal of Differential Geometry* 39:1 (1994), 57–63. MR1258914 (95c:58137)
9. Boris Hasselblatt and Amie Wilkinson. ”Prevalence of Non-Lipschitz Anosov Foliations.” *Ergodic Theory and Dynamical Systems* 19:3 (1999), 643–656. MR1695913 (2000f:37035)
10. Morris Hirsch amd Charles C. Pugh. ”Smoothness of Horocycle Foliations.” *Journal of Differential Geometry* 10 (1975), 225–238. MR0368077 (51 #4319)
11. Morris Hirsch, Charles C. Pugh, and Michael Shub. *Invariant Manifolds*. Lecture Notes in Mathematics 583. Berlin: Springer-Verlag, 1977.
12. Eberhard Hopf. ”Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung.” *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Physikalische Klasse* 91 (1939), 261–304. MR0001464 (1,243a)
13. Anatole Katok and Boris Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Encyclopedia of Mathematics and Its Applications 54. Cambridge, UK: Cambridge University Press, 1995. MR1326374 (96c:58055)
14. Wilhelm Klingenberg. *Riemannian Geometry*. Berlin: de Gruyter, 1982. MR0666697 (84j:53001)
15. Vojislav Marić. *Regular Variation and Differential Equations*. Lecture Notes in Mathematics

1726. Berlin: Springer-Verlag, 2000. MR1753584 (2001m:34120)
16. Edward Omey. ”Rapidly Varying Behaviour of the Solutions of a Second Order Linear Differential Equation.” In *Proceedings of the Seventh International Colloquium on Differential Equations*, pp. 295–303. Utrecht: VSP, 1997. MR1466007 (98j:34055)
17. Gabriel Pedro Paternain. *Geodesic Flows.* Progress in Mathematics 180. Boston-Basel-Berlin: Birkhäuser, 1999. MR1712465 (2000h:53108)
18. George Neville Watson. *A Treatise on the Theory of Bessel Functions.* Cambridge, UK: Cambridge University Press, 1922, 1944, 1966, 1995. MR0010746 (6,64a)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1995704 (2004f:37001) 37-01 (37-02)
Hasselblatt, Boris (1-TUFT); **Katok, Anatole** (1-PAS-CDY)

★**A first course in dynamics.**
With a panorama of recent developments.
Cambridge University Press, New York, 2003. *x*+424 *pp.* $95.00; $35.00 paperbound.
ISBN 0-521-58304-7; 0-521-58750-6

The book consists of two parts. Part 1 is a rather advanced, one semester, undergraduate course in dynamical systems; Part 2 gives an overview of several major subfields of dynamical systems. The course contained in Part 1 stands out among all other existing undergraduate dynamical systems texts in that it gives a mathematically thorough introduction to a very wide variety of subareas and concepts of dynamics. It presents both the topological and statistical (ergodic) point of view. Part 1 starts with basic examples, notions and problems (Chapters 1–2). Low-level multivariate calculus and an elementary course in ordinary differential equations are sufficient prerequisites for this portion of the course. The difficulty level gradually increases through Chapters 3–5 which discuss simple systems with stable behavior, homeomorphisms of the circle, Cantor sets, equidistribution of orbits for circle rotations and translations of the torus. These chapters assume familiarity with linear algebra which is somewhat beyond an elementary (freshman or sophomore course). The remaining chapters of the course (6–8, and possibly 9) are much more demanding both in terms of the necessary prerequisites (advanced calculus and linear algebra, elements of point set topology, etc.) and in the level of mathematical sophistication. This portion of the course is aimed at a very advanced and quite motivated undergraduate student. It includes basics of conservative systems, topological dynamics, symbolic dynamics, topological entropy and chaotic behavior. As the authors say in the preface, Part 2 (Panorama of dynamical systems) provides applications of ideas from Part 1 and connects them to contemporary topics of research. Most of this material should be presented only to an elite undergraduate student. The Panorama starts with applications of the contraction mapping principle: the implicit and inverse function theorems, foundations of ordinary differential equations (existence and uniqueness theorems, dependence on initial values and parameters), and the stable manifold theorem for a hyperbolic fixed point. It proceeds through a very wide range of topics—from foundations of hyperbolic dynamics, the Feigenbaum phenomenon and strange attractors, through variational methods for twist maps and closed geodesics on the sphere, to Weyl's theorem on the distribution of values of polynomials and other important applications of dynamical systems to number theory. The authors are to a large degree successful in decreasing the awkwardness of avoiding the concepts of a Riemannian manifold, Lebesgue measure and integration and some other notions which necessarily appear in a graduate dynamical systems course. I highly recommend this book for an advanced undergraduate course in dynamical systems. (For a more detailed and advanced text, see A. B. Katok and B. Hasselblatt [*Introduction to the modern theory of dynamical systems*, Cambridge Univ. Press, Cambridge, 1995; MR1326374 (96c:58055)]).

Reviewed by *M. I. Brin*

MR1975364 (2004i:37039) 37C05 (37C15 37G05)
Guysinsky, Misha (1-PAS); **Hasselblatt, Boris** (1-TUFT); **Rayskin, Victoria** (1-UCLA)
Differentiability of the Hartman-Grobman linearization. (English summary)
Discrete Contin. Dyn. Syst. **9** (2003), *no. 4,* 979–984.

The Hartman-Grobman theorem asserts that a C^1 diffeomorphism can be C^0 linearized near a hyperbolic fixed point, say $0 \in \mathbb{R}^n$. Hartman also showed that in $\mathbb{R}^2$ the linearization is C^1. However, he also gave an example in $\mathbb{R}^3$ for which there is no C^1 linearization.

In order to use the linearization to understand the behaviour near a hyperbolic fixed point, one needs the linearization to be smoother than C^0.

The authors prove the following theorem: Theorem. If the map of the Hartman-Grobman theorem is C^∞, then the linearizing homeomorphism is differentiable at the origin, and its derivative at 0 is the identity.

The authors' proof is "inspired" by normal form methods. They need the C^∞ requirement to apply a theorem by Bronstein and Kopanskiĭ to "reduce the map to its second-order normal form expansion via a C^1 conjugacy". Other than that, the authors believe that their result should hold for a C^2 map.

Reviewed by *Mohamed Sami ElBialy*

MR1934140 (2003h:37001) 37-03 (01A60 01A70 70-03)
Hasselblatt, Boris (1-TUFT); **Katok, Anatole** (1-PAS)
The development of dynamics in the 20th century and the contribution of Jürgen Moser. (English summary)
Ergodic Theory Dynam. Systems **22** (2002), *no. 5,* 1343–1364.

J. K. Moser (1928–1999) made profound and fundamental contributions to many branches of mathematics. The present paper surveys his work in dynamics and related areas. For brief accounts of all the achievements of Moser in mathematics, the authors recommend the notes by E. J. Zehnder [Jahresber. Deutsch. Math.-Verein. **95** (1993), no. 2, 85–94; MR1218340 (94d:01080)] and J. N. Mather et al. [Notices Amer. Math. Soc. **47** (2000), no. 11, 1392–1405; MR1794131 (2001h:01054)].

The first part of the paper outlines the development of dynamics from Newton and Laplace to the contemporary theories, with an emphasis on the 20th century, and provides general characteristics of the topics and style of Moser's work. Here are some quotations: "Always keenly interested in the work of others, he was able to discern the fundamental trends and invariably made essential, often fundamental, contributions." "We cannot think of another mathematician in the period after 1960 who had such a broad view and comprehensive understanding of virtually all major trends in dynamics and influenced their development to a similar degree." "In his work he usually searched for wisdom rather than simply knowledge, and thus he strongly emphasized developments of methods and insights over pushing a specific result to the limit." "The leading theme of virtually all of Moser's work in dynamics is the search for elements of stable behavior in dynamical systems with respect to either initial conditions or perturbations of the system." In the second part of the paper, the authors discuss (unavoidably omitting many details) Moser's contributions to the KAM (Kolmogorov-Arnol′d-Moser) theory, the Aubry-Mather theory, completely integrable Hamiltonian systems, and hyperbolic dynamics, as well as some separate results. The exposition is very vivid and provides both the mathematical ideas and historical accounts. However, it is hardly intelligible to a layman. The bibliography contains 91 references, including 29 works by Moser.

Reviewed by *Mikhail B. Sevryuk*

References

1. *Smooth Ergodic Theory and its Applications (Summer Research Institute Seattle, WA, 1999) (Proc. Symp. in Pure Mathematics, 69)*. Eds. A. Katok, R. de la Lave, Y. Pesin and H. Weiss. American Mathematical Society, Providence, RI, 2001. MR1858533 (2002d:37005)
2. D. V. Anosov. *Geodesic Flows on Riemannian Manifolds with Negative Curvature (Proc. Steklov Institute of Mathematics, 90)*. American Mathematical Society, Providence, RI, 1967. MR0224110 (36 #7157)
3. D. V. Anosov and A. B. Katok. New examples in smooth ergodic theory. Ergodic diffeomor-

phisms. *Trans. Moscow Math. Soc.* **23** (1970), 1–35. MR0370662 (51 #6888)

4. V. I. Arnold. Small denominators. I. Mapping the circle onto itself. *Izv. Akad. Nauk SSSR Ser. Mat.* **25** (1961), 21–86; Correction to V. Arnold's paper: 'Small denominators. I.'. *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 479–480. MR0164049 (29 #1348)
5. V. I. Arnold. Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian. *Usp. Mat. Nauk* **18**(5) (1963), 13–40. MR0163025 (29 #328)
6. V. I. Arnold. Small denominators and problems of stability of motion in classical and celestial mechanics. *Usp. Mat. Nauk* **18**(6) (1963), 91–192. MR0170705 (30 #943)
7. E. Artin. Ein mechanisches System mit quasiergodischen Bahnen. *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität* **3** (1924), 170–175.
8. V. Bangert. *Mather Sets for Twist Maps and Geodesics on Tori (Dynamics Reported, 1)*. Wiley, Chichester, 1988, pp. 1–56. MR0945963 (90a:58145)
9. J. Barrow-Green. *Poincaré and the Three Body Problem (American Mathematical Society/London Mathematical Society History of Mathematics, 11)*. American Mathematical Society, Providence, RI, 1997. MR1415387 (97g:01013)
10. G. D. Birkhoff. Proof of the ergodic theorem. *Proc. Natl. Acad. Sci. USA* **17** (1931), 656–660.
11. G. D. Birkhoff. *Dynamical Systems (Colloquium Publications, 9)*. American Mathematical Society, Providence, RI, 1927 MR0209095 (35 #1)
12. G. D. Birkhoff. *Nouvelles recherches sur les systèmes dynamique*, 'Memoriae Pont. Acad. Sci. Novi Lyncaei, s. 3, Vol. 1, 1935, pp. 85–216' according to the collected works.
13. M. L. Cartwright. Forced oscillations in nonlinear systems. *Contributions to the Theory of Nonlinear Oscillations (Annals of Mathematics Studies, 20)*. Princeton University Press, Princeton, NJ, 1950, pp. 149–241. MR0035355 (11,722a)
14. M. L. Cartwright and J. E. Littlewood. On non-linear differential equations of the second order. I. The equation $\ddot{y} - k(1-y^2)y + y = b\lambda k\cos(\lambda t + a)$, k large. *J. London Math. Soc.* **20** (1945), 180–189. MR0016789 (8,68g)
15. C. Conley. *Isolated Invariant Sets and The Morse Index (CBMS Regional Series in Mathematics, 38)*. American Mathematical Society, Providence, RI, 1978. MR0511133 (80c:58009)
16. C. Conley and E. Zehnder. Morse-type index theory for flows and periodic solutions for Hamiltonian equations. *Commun. Pure Appl. Math.* **37**(2) (1984), 207–253. MR0733717 (86b:58021)
17. P. M. M. Duhem. *La Théorie Physique: Son Objet, Sa Structure*. Marcel Rivière & Cie., Paris, 1906, 1914; Engl. Transl. *The Aim and Structure of Physical Theory*. Princeton Science Library, Princeton University Press, 1991.
18. E. Fermi, J. Pasta and S. Ulam. Studies of nonlinear problems. I. *Los Alamos Report, LA-1940*, 1955. Reproduced in *Nonlinear Wave Motion (Proc. of the Summer Seminar, sponsored by the American Mathematical Society and the Society for Industrial and Applied Mathematics, Clarkson College of Technology, Potsdam, NY, 1972) (Lectures in Applied Mathematics, 15)*. Ed. A. C. Newell. American Mathematical Society, Providence, RI, 1974, pp. 143–156. MR0336014 (49 #790)
19. C. S. Gardner, M. D. Kruskal and R. M. Miura. Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. *J. Math. Phys.* **9** (1968), 1204–1209.

MR0252826 (40 #6042b)

20. C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura. Korteweg-de Vries equation and generalization. VI. Methods for exact solution. *Commun. Pure Appl. Math.* **27** (1974), 97–133. MR0336122 (49 #898)
21. J. S. Hadamard. Les surfaces à courbures opposées et leurs lignes géodesiques. *J. Math. Pures Appliquées* **4**(5) (1898), 27–73.
22. B. Hasselblatt. Regularity of the Anosov splitting and of horospheric foliations. *Ergod. Th. & Dynam. Sys.* **14** (1994), 645–666. MR1304137 (95j:58130)
23. G. A. Hedlund. Geodesics on a two-dimensional Riemannian manifold with periodic coefficients. *Ann. Math.* **33** (1932), 719–739. MR1503086
24. G. A. Hedlund. Fuchsian groups and transitive horocycles. *Duke Math. J.* **2** (1936), 530–542. MR1545946
25. G. A. Hedlund. Fuchsian groups and mixtures. *Ann. Math.* (2) **40**(2) (1939), 370–383. MR1503464
26. M. R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Inst. Hautes Études Sci. Publ. Math.* **49** (1979), 5–233. MR0538680 (81h:58039)
27. H. Hofer and E. Zehnder. *Symplectic Invariants and Hamiltonian Dynamics*. Birkhäuser, 1994. MR1306732 (96g:58001)
28. E. Hopf. *Fuchsian Groups and Ergodic Theory (Transactions of the American Mathematical Society, 39)*. American Mathematical Society, Providence, RI, 1936, pp. 299–314. MR1501848
29. E. Hopf. *Ergodentheorie (Ergebnisse der Mathematik, 5)*. Springer, 1937.
30. E. Hopf. Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, Mathematisch-Physikalische Klasse* **91** (1939), 261–304. MR0001464 (1,243a)
31. R. Johnson and J. K. Moser. The rotation number for almost periodic potentials. *Comm. Math. Phys.* **84**(3) (1982), 403–438; Erratum: ‘The rotation number for almost periodic potentials’. *Comm. Math. Phys.* **90**(2) (1983), 317–318. MR0714441 (85a:34032)
32. A. B. Katok and K. Burns. Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems. *Ergod. Th. & Dynam. Sys.* **14** (1994), 757–785. MR1304141 (95j:58097)
33. A. B. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995. MR1326374 (96c:58055)
34. P. Koebe. Riemannsche Mannigfaltigkeiten und nicht euklidische Raumformen. *Sitzungsberichte der Preußischen Akademie der Wissenschaften, I*, (1927) 164–196; *II, III*, (1928) 345–442; *IV*, (1929) 414–557; *V, VI*, (1930) 304–364, 504–541; *VII*, (1931) 506–534.
35. A. N. Kolmogorov. On conservation of conditionally periodic motions for a small change in Hamilton’s function. *Dokl. Akad. Nauk SSSR (N.S.)* **98** (1954), 527–530; Engl. Transl. *Stochastic Behavior in Classical and Quantum Hamiltonian Systems (Volta Memorial Conference, Como, 1977) (Lecture Notes in Physics, 93)*. Springer, 1979. MR0068687 (16,924c)
36. A. N. Kolmogorov. Théorie générale des systèmes dynamiques et mécanique classique. *Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1*. Noordhoff, Groningen; North-Holland, Amsterdam, 1957, pp. 315–333; Engl. Transl. R. Abraham.

Foundations of Mechanics. Benjamin, 1967, Appendix D. MR0097598 (20 #4066)
37. N. Kryloff and N. Bogoliouboff. La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire. *Ann. Math.* **38**(1) (1937), 65–113. MR1503326
38. P. Simon marquis de Laplace. *Traité de mécanique céleste*, 1799–1825; Engl. Transl. N. Bowditch. Laplace's Mécanique Céleste. Reprint Services Corp.
39. P. Simon marquis de Laplace. *Philosophical Essay on Probabilities*. Translated from the 5th French edition of 1925 by A. I. Dale. Springer, New York, 1995. MR1325241 (96c:01028)
40. M. Levi and J. K. Moser. A Lagrangian proof of the invariant curve theorem for twist mappings. *Smooth Ergodic Theory and its Applications (Summer Research Institute Seattle, WA, 1999) (Proc. Symp. in Pure Mathematics, 69)*. Eds. A. Katok, R. de la Lave, Y. Pesin and H. Weiss. American Mathematical Society, Providence, RI, 2001, pp. 733–746. MR1858552 (2002h:37066)
41. P. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Commun. Pure Appl. Math.* **21** (1968), 467–490. MR0235310 (38 #3620)
42. R. de la Llave. A tutorial on KAM theory. *Smooth Ergodic Theory and its Applications (Summer Research Institute Seattle, WA, 1999) (Proc. Symp. in Pure Mathematics, 69)*. Eds. A. Katok, R. de la Lave, Y. Pesin and H. Weiss. American Mathematical Society, Providence, RI, 2001, pp. 175–292. MR1858536 (2002h:37123)
43. F. Löbell. Über die geodätischen Linien der Clifford-Kleinschen Flächen. *Math. Z.* **30** (1929), 572–607. MR1545081
44. J. N. Mather. Action minimizing invariant measures for positive definite Lagrangian systems. *Math. Z.* **207** (1991), 169–207. MR1109661 (92m:58048)
45. J. N. Mather, H. P. McKean, L. Nirenberg and P. H. Rabinowitz. Jürgen K. Moser (1928–1999). *Notices Amer. Math. Soc.* **47**(11) (2000), 1392–1405. MR1794131 (2001h:01054)
46. J. N. Mather and G. Forni. Action minimizing orbits in Hamiltonian systems. *Transition to Chaos in Classical and Quantum Mechanics (Montecatini Terme, 1991) (Lecture Notes in Mathematics, 1589)*. Springer, Berlin, 1994, pp. 92–186. MR1323222 (97a:58059)
47. M. Morse. Recurrent geodesics on a surface of negative curvature. *Trans. Amer. Math. Soc.* **22** (1921), 84–100. MR1501161
48. M. Morse. A fundamental class of geodesics on any closed surface of genus greater than one. *Trans. Amer. Math. Soc.* **26** (1924), 25–61. MR1501263
49. J. K. Moser. The analytic invariants of an area-preserving mapping near a hyperbolic fixed point. *Commun. Pure Appl. Math.* **9** (1956), 673–692. MR0086981 (19,278b)
50. J. K. Moser. A new technique for the construction of solutions of nonlinear differential equations. *Proc. Natl. Acad. Sci. USA* **47** (1961), 1824–1831. MR0132859 (24 #A2695)
51. J. K. Moser. On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen Math.-Phys.* Kl. II **1962** (1962), 1–20; Remark on the paper: 'On invariant curves of area-preserving mappings of an annulus'. *Regul. Chaotic Dyn.* **6**(3) (2001), 337–338. MR1860151 (2002g:37080)
52. J. K. Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.* **120** (1965), 286–294. MR0182927 (32 #409)

53. J. K. Moser. A rapidly convergent iteration method and non-linear partial differential equations. I. *Ann. Scuola Norm. Sup. Pisa* **20**(3) (1966), 265–315; A rapidly convergent iteration method and non-linear differential equations. II. *Ann. Scuola Norm. Sup. Pisa* **20**(3) (1966), 499–535. MR0206461 (34 #6280)
54. J. K. Moser. Convergent series expansions for quasi-periodic motions. *Math. Ann.* **169** (1967), 136–176. MR0208078 (34 #7888)
55. J. K. Moser. *Lectures on Hamiltonian Systems (Mem. Amer. Math. Soc., 81)*. American Mathematical Society, Providence, RI, 1968. MR0230498 (37 #6060)
56. J. K. Moser. On a theorem of Anosov. *J. Diff. Eqns* **5** (1969), 411–440. MR0238357 (38 #6633)
57. J. K. Moser. *Stable and Random Motions in Dynamical Systems (Ann. Math. Stud., 77)*. Princeton University Press, Princeton, NJ, 1973; Reprint, Princeton University Press, Princeton, NJ, 2001. MR0442980 (56 #1355)
58. J. K. Moser. A lemma on hyperbolic fixed points of diffeomorphisms. *Usp. Mat. Nauk* **29**(2) (1974), 228–232; Engl. Transl. *Russian Mathematical Surveys* **29**(2) (1974), 236–241. MR0423421 (54 #11400)
59. J. K. Moser. Three integrable Hamiltonian systems connected with isospectral deformations. *Surveys in Applied Mathematics (Proc. First Los Alamos Symp. Math. in Natural Sci., Los Alamos, NM, 1974)*. Academic Press, New York, 1976, pp. 235–258. MR0665768 (58 #32198)
60. J. K. Moser. Finitely many mass points on the line under the influence of an exponential potential—an integrable system. *Dynamical systems, Theory and Applications (Rencontres, Battelle Res. Inst., Seattle, 1974) (Lecture Notes in Physics, 38)*. Springer, Berlin, 1975, pp. 467–497. MR0455038 (56 #13279)
61. J. K. Moser. Stable and unstable motion in dynamical systems. *Nonlinear Dynamics and the Beam-beam Interaction (Symp., Brookhaven Nat. Lab., New York, 1979)*. American Institute of Physics, New York, 1980, pp. 222–235. MR0624985 (83b:58030)
62. J. K. Moser. *Geometry of Quadrics and Spectral Theory (The Chern Symposium, 1979)*. Springer, 1980, pp. 147–188. MR0609560 (82j:58064)
63. J. K. Moser. Various aspects of integrable Hamiltonian systems. *Dynamical Systems (Bressanone, 1978)*. Liguori, Naples, 1980, pp. 137–195; also published in *Dynamical Systems (C.I.M.E. Summer School, Bressanone, 1978) (Progress in Mathematics, 8)*. Birkhäuser, Boston, 1980, pp. 233–289. MR0589592 (83a:58042a)
64. J. K. Moser. *Integrable Hamiltonian Systems and Spectral Theory*. Lezioni Fermiane, Scuola Normale Superiore, Pisa, 1983. MR0854865 (87j:58042)
65. J. K. Moser. Break-down of stability. *Nonlinear Dynamics Aspects of Particle Accelerators (Santa Margherita di Pula, 1985) (Lecture Notes in Physics, 247)*. Springer, Berlin, 1986, pp. 492–518. MR0846252 (87m:58061)
66. J. K. Moser. A stability theorem for minimal foliations on a torus. *Ergod. Th. & Dynam. Sys.* **8*** (1988), Charles Conley Memorial Issue, 251–281. MR0967641 (90h:58018)
67. J. K. Moser. Minimal solutions of a variational problem on a torus. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3**(3) (1986), 229–272. MR0847308 (88a:58058)
68. J. K. Moser. On commuting circle mappings and simultaneous Diophantine approximations. *Math. Z.* **205**(1) (1990), 105–121. MR1069487 (92e:58120)

69. J. K. Moser. Minimal foliations on a torus. *Lecture Notes in Mathematics, vol. 1365*. Springer, 1989, pp. 62–69. MR0994019 (90f:58153)
70. J. K. Moser and A. P. Veselov. Discrete versions of some classical integrable systems and factorization of matrix polynomials. *Commun. Math. Phys.* **139**(2) (1991), 217–243. MR1120138 (92g:58054)
71. J. K. Moser. An unusual variational problem connected with Mather's theory for monotone twist mappings. *Seminar on Dynamical Systems (St. Petersburg, 1991)*. Birkhäuser, Basel, 1994, pp. 81–89. MR1279390 (95g:58198)
72. J. K. Moser. Smooth approximation of Mather sets of monotone twist mappings. *Commun. Pure Appl. Math.* **47**(5) (1994), 625–652. MR1278347 (95g:58199)
73. J. K. Moser. Old and new application of KAM theory. *Hamiltonian Systems with Three or More Degrees of Freedom (NATO Adv, Sci. Inst. Ser. Math Phys. Sci., 533)*. Kluwer, 1999, pp. 184–192. MR1720893 (2000g:37092)
74. J. K. Moser. Recollections. Concerning the early development of KAM theory. *The Arnoldfest (Toronto, ON, 1997)*. American Mathematical Society, Providence, RI, 1999, pp. 19–21. MR1733565
75. C. G. Neumann. De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur. *J. Reine Angew. Math.* **56** (1859), 46–63.
76. J. von Neumann. Proof of the quasi-ergodic hypothesis. *Proc. Natl. Acad. Sci. USA* **18** (1932), 70–82.
77. J. von Neumann. Zur Operatorenmethode in der klassischen Mechanik. *Ann. Math.* **33** (1932), 587–642. MR1503078
78. I. Newton. *Philosophiæ Naturalis Principia Mathematica*, 3rd edn. Will & John Innys, London, 1729; Translation and Guide: *The Principia, Mathematical Principles of Natural Philosophy*. University of California Press, 1999.
79. J. H. Poincaré. Sur le problème des trois corps et les equations de la dynamique. *Acta Mathematica* **13** (1890), 1–270.
80. J. H. Poincaré. *Les méthodes nouvelles de la mécanique céleste*. Paris, 1892–1899; Engl. Transl. *New Methods of Celestial Mechanics (History of Modern Physics and Astronomy, 13)*. Ed. D. Goroff. American Institute of Physics, New York, 1993.
81. B. van der Pol and J. van der Mark. Frequency demultiplication. *Nature* **120** (1927), 363–364.
82. P. H. Rabinowitz. Periodic solutions of Hamiltonian systems. *Commun. Pure Appl. Math.* **31** (1978), 157–184. MR0467823 (57 #7674)
83. D. Salamon and E. Zehnder. KAM theory in configuration space. *Comment. Math. Helv.* **64** (1989), 84–132. MR0982563 (90d:58045)
84. H. Seifert. Periodische BEWEGUNGEN mechanischer Systeme. *Math. Z.* **51** (1948), 197–216. MR0025693 (10,49d)
85. C. L. Siegel. Iteration of analytic functions. *Ann Math.* (2) **43** (1942), 607–612. MR0007044 (4,76c)
86. C. L. Siegel and J. K. Moser. *Lectures on Celestial Mechanics*. Springer, New York, 1971; Reprint of the 1971 translation. Springer, Berlin, 1995. MR1345153 (96d:70002)
87. S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

MR0228014 (37 #3598)

88. J.-C. Yoccoz. Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne. *Ann. Sci. École Norm. Sup.* (4) **17**(3) (1984), 333–359. MR0777374 (86j:58086)
89. J.-C. Yoccoz. Introduction to hyperbolic dynamics. *Real and Complex Dynamical Systems (Proceedings of the NATO Advanced Study Institute held in Hillerød, June 20-July 2, 1993) (NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, 464)*. Eds. B. Branner and P. Hjorth. Kluwer, Dordrecht, 1995, pp. 265–291. MR1351526 (96h:58131)
90. E. Zehnder. Generalized implicit function theorems with applications to small divisors problems. I. *Commun. Pure Appl. Math.* **28** (1975), 91–140. MR0380867 (52 #1764)
91. E. Zehnder. Cantor-Medaille für Jürgen Moser. *Jahresber. Deutsch. Math.-Verein.* **95**(2) (1993), 85–94. MR1218340 (94d:01080)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1928520 (2004b:37047) 37Dxx (37Cxx)

Hasselblatt, Boris (1-TUFT)

Hyperbolic dynamical systems.

Handbook of dynamical systems, Vol. 1A, 239–319, *North-Holland, Amsterdam*, 2002.

The aim of this survey is to outline important results in the theory of uniformly hyperbolic dynamical systems on compact spaces, as well as their extension to nonuniformly hyperbolic systems. No proofs are provided; the arguments are sometimes sketched.

The main part deals with the study of uniformly hyperbolic diffeomorphisms and flows, both from the local and global viewpoints. It begins with the theory of stable and unstable laminations, with an emphasis on their degree of regularity. The author then presents the basic topological properties of the dynamics: expansivity, shadowing of orbits, and spectral decomposition. This leads to the problem of structural stability, and topological classification of Anosov transformations and flows. Whereas an Anosov diffeomorphism is structurally stable, which means topologically conjugate to any small perurbation, the conjugacy cannot be expected to be smooth: indeed a smooth conjugacy would preserve lengths of periodic orbits, and also the regularity of the stable and unstable distributions. The first obstruction is related to cohomology equations, the Lyapunov cocycle and the Livšic theorem. The second obstruction leads to smooth rigidity results, which say roughly that only smooth deformations of systems of algebraic origin can have stable/unstable foliations with smooth holonomy. The author reviews these two aspects.

Zeta functions, growth of periodic orbits, Markov partitions and questions related to Gibbs measures and thermodynamical formalism are only briefly mentioned.

The last part deals with the nonuniform theory: Lyapunov exponents, the Oseledets theorem, regular neighborhoods and hyperbolic measures are presented with some details. The survey then focuses on surfaces. Standard results from the uniform theory hold in the nonuniform setting, as soon as there are no zero Lyapunov exponents: the absolute continuity of the invariant foliations, the shadowing lemma, the Livšic theorem for example; also, if the entropy of the hyperbolic measure is nonzero, there exist embedded horseshoes with entropy arbitrarily close to the entropy of the measure.

{For the entire collection see MR1928517 (2003c:37002)}

Reviewed by *Yves Coudene*

MR1928518 (2004c:37001) 37-XX
Hasselblatt, Boris (1-TUFT); **Katok, Anatole** (1-PAS)
Principal structures.
Handbook of dynamical systems, Vol. 1*A*, 1–203, *North-Holland, Amsterdam*, 2002.

From the introduction: “Dynamical systems has grown from various roots into a field of great diversity that interacts with many branches of mathematics as well as with the sciences. The purpose of this survey is to describe the general framework for several principal areas of the theory of dynamical systems. We are aware that this is an ambitious goal and that the presentation is bound to be both brief and in many respects superficial.

“Our primary aim is to set the stage for the surveys collected in this and the subsequent volume by establishing the unity of the various specialities within dynamics. The range of surveys in these volumes therefore has a strong effect on the presentation given here. Certain topics, which appear in a number of surveys and which we consider as basic for several branches of dynamics, are presented in some detail. Examples are recurrence in topological dynamics, ergodicity, topological and metric entropy, the variational principle for entropy, invariant stable and unstable manifolds, and cocycles over dynamical systems. Even such topics are usually discussed with only a few complete proofs. Topics central to any of the subsequent surveys are often discussed just enough to place them in the greater context, deferring to the corresponding survey for exact statements and further detail. Examples of these are dynamical ζ-functions, variational methods in Lagrangian and Hamiltonian dynamics, KAM theory, dynamics of unipotent homogeneous systems, and dynamical methods in combinatorial number theory. Nevertheless, some topics are surveyed here because they play an essential role in the overall picture, even though they are not given much attention in subsequent surveys. Bifurcations and applications per se are virtually absent here because they are in the purview of other volumes in the series.

“A possible use of this survey is as an introduction to mathematicians unfamiliar with dynamics, and it may be interesting to experts as an overview of a diverse field. With this in mind we pay attention to examples, motivations, informal explanations and discussion of key special cases or simplified versions of general results. Nevertheless, they may often be too brief and may sometimes look cryptic to a nonexpert reader. Expanding the pedagogical aspects of the survey substantially would interfere with its primary goal and expand its size beyond a reasonable limit. Hopefully, a compromise between comprehensiveness and accessibility has been achieved.

“A limited number of key results are proved in the text, when the importance of the result, the insights provided by the proof, or its brevity suggested doing so. Other results are provided with sketches or outlines of proofs; many more are only formulated or just mentioned.

“The structure of this survey is intended to reflect a coherent framework. Accordingly, this chapter introduces a collection of important notions in generic terms, i.e., without relying on any specific structure of the dynamical system (topological, measure-theoretic, smooth, etc.). Although examples are therefore deferred, this serves to provide a structure that organizes the notions and techniques in such a way that later chapters can present large subareas of dynamics in a coherent fashion. Starting from Section 2 we introduce basic examples as close to the beginning of each chapter as practicable and then intersperse further examples, as well as comments on previously introduced ones, throughout the chapters. The central structural elements are presented in the following order: the notions of equivalence, principal constructions, recurrence, and orbit growth. The chapters on topological dynamics and ergodic theory follow this pattern closely. The succeeding chapters on smooth dynamics, which are based on the earlier ones, fit into the same framework as well, although the starting point and emphasis are, of necessity, slightly different. Some background material is incorporated into the text. Examples of these are the treatment of Lebesgue spaces, symplectic manifolds, and Hamiltonian formalism.

“The bibliography is divided into several parts, beginning with the surveys in this volume and some of those planned for the companion volume. In order to help choose further reading we provide a list of major sources. Of these we most frequently refer to our book [*Introduction to the modern theory of dynamical systems*, Cambridge Univ. Press, Cambridge, 1995; MR1326374 (96c:58055)] for proofs omitted here. A particularly valuable recent publication is the collection [*Smooth ergodic theory and its applications (Seattle, WA, 1999)*, Proc. Sympos. Pure Math., 69, Amer. Math. Soc., Providence, RI, 2001; MR1858533 (2002d:37005)], which contains numerous expository papers on a broad variety of subjects treated in these volumes. Six volumes of the Springer Encyclopaedia of Mathematical Sciences treat dynamical systems in the narrow sense. They constitute an expository project that is similar in size to the present series of volumes and complements it in many respects.

“While most of the remaining bibliography entries are referred to, often for specific results or topics, some are included for added convenience. We do not claim to present a comprehensive or even fully representative bibliography on any of the topics. The bibliographies in subsequent surveys and in our major sources are better suited for that purpose. Furthermore, we are aware of the bias in the references toward works that fit into our own point of view on the subject, as well as the omission of some important sources with which we are not sufficiently familiar.”

{For the entire collection see MR1928517 (2003c:37002)}

MR1928517 (2003c:37002) 37-06

★**Handbook of dynamical systems. Vol. 1A.**

Edited by B. Hasselblatt and A. Katok.

North-Holland, Amsterdam, 2002. *xii*+1220 *pp*. $180.00. *ISBN* 0-444-82669-6

Contents: Boris Hasselblatt and Anatole Katok, Principal structures (1–203); Jean-Paul Thouvenot, Entropy, isomorphism and equivalence in ergodic theory (205–238); Boris Hasselblatt, Hyperbolic dynamical systems (239–319); N. Chernov [Nikolai I. Chernov], Invariant measures for hyperbolic dynamical systems (321–407); Mark Pollicott, Periodic orbits and zeta functions (409–452); Gerhard Knieper, Hyperbolic dynamics and Riemannian geometry (453–545); John Franks and Michał Misiurewicz, Topological methods in dynamics (547–598); M. Jakobson and G. Świątek, One-dimensional maps (599–664); Renato Feres and Anatole Katok, Ergodic theory and dynamics of G-spaces (with special emphasis on rigidity phenomena) (665–763); Douglas Lind and Klaus Schmidt [Klaus Schmidt[1]], Symbolic and algebraic dynamical systems (765–812); Dmitry Kleinbock, Nimish Shah [Nimish A. Shah] and Alexander Starkov [A. N. Starkov], Dynamics of subgroup actions on homogeneous spaces of Lie groups and applications to number theory (813–930); Alex Furman, Random walks on groups and random transformations (931–1014); Howard Masur and Serge Tabachnikov [Sergei Tabachnikov], Rational billiards and flat structures (1015–1089); P. H. Rabinowitz, Variational methods for Hamiltonian systems (1091–1127); H. Hofer [Helmut H. W. Hofer], K. Wysocki and E. Zehnder, Pseudoholomorphic curves and dynamics in three dimensions (1129–1188).

{The papers are being reviewed individually.}

MR1920652 (2003g:37044) 37D10 (37D20 37J10)
Hasselblatt, Boris (1-TUFT)
Critical regularity of invariant foliations. (English summary)
Discrete Contin. Dyn. Syst. **8** (2002), *no. 4,* 931–937.

The author studies the regularity of the unstable subbundle of an Anosov diffeomorphism f that preserves a symplectic form. It is shown that there is an open set of such diffeomorphisms for which either the unstable bundle is C^ε only on a negligible set, or it is better than C^1.

The differential of the transformation f acts on the space of bounded sections of the tangent bundle by $Df \circ X \circ f^{-1}$. The spectrum of this operator is called the Mather spectrum of f.

Consider numbers $\overline{\mu}_f, \mu_f, \overline{\mu}_s, \mu_s$ with $0 < \overline{\mu}_f < \mu_f < \overline{\mu}_s < \mu_s < 1$, and suppose that there exist α, η in $(0,1)$ with $(\mu_f)^\alpha < (\mu_s)^2$ and $\overline{\mu}_s\mu_f < (\overline{\mu}_f)^{1+\eta}$. Then a symplectic Anosov diffeomorphism, with a Mather spectrum contained in the rings $\overline{\mu}_f < |z| < \mu_f$ and $\overline{\mu}_s < |z| < \mu_s$ and their inverse, is either C^α only on a negligible set, or $C^{1+\eta}$.

The proof relies on the study of an obstruction defined in terms of the matrix of Df in adapted coordinates.

Reviewed by *Yves Coudene*

MR1695913 (2000f:37035) 37D20 (37C20 37D05 37D30 37J05)
Hasselblatt, Boris (1-TUFT); **Wilkinson, Amie** (1-NW)
Prevalence of non-Lipschitz Anosov foliations. (English summary)
Ergodic Theory Dynam. Systems **19** (1999), *no. 3,* 643–656.

The authors give sharp regularity results for the invariant subbundles of hyperbolic dynamical systems in terms of contraction and expansion rates. They also prove optimality of their results, in a strong sense, by constructing open dense sets of codimension one systems where this regularity is not exceeded. Another goal of the paper is to exhibit open dense sets of symplectic, geodesic and codimension one systems where the estimates for the Hölder exponents of the holonomy maps found in a paper by C. C. Pugh, M. Shub and A. Wilkinson [Duke Math. J. **86** (1997), no. 3, 517–546; MR1432307 (97m:58155)] are optimal. Yet another goal is to produce open sets of symplectic Anosov diffeomorphisms and flows with low transverse Hölder regularity of the invariant foliations almost everywhere. A corollary of this result is the prevalence of low regularity for conjugacies between two Anosov systems. Finally, a new connection between the transverse regularity of foliations and their tangent subbundles is established. All these results might have applications in the future study of stochastic properties of hyperbolic and partially hyperbolic systems.

Reviewed by Viorel Niţică

References

1. D. V. Anosov. Geodesic flows on Riemann manifolds with negative curvature. *Proc. Steklov Inst.* **90** (1967). MR0224110 (36 #7157)
2. D. V. Anosov. Tangent fields of transversal foliations in 'U-systems'. *Math. Notes Acad. Sci. USSR* **2**(5) (1967), 818–823. MR0242190 (39 #3523)
3. V. I. Arnol'd. *Mathematical Methods of Classical Mechanics*. Springer, Berlin, 1979. MR0542447 (80h:70001)
4. L. Barreira. A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems. *Ergod. Th. & Dynam. Sys.* **16**(5) (1996), 871–927. MR1417767 (98a:58124)
5. Y. Benoist, P. Foulon and F. Labourie. Flots d'Anosov à distributions stable et instable différentiables. *J. Amer. Math. Soc.* **5**(1) (1992), 33–74. MR1124979 (93b:58112)
6. G. Besson, G. Courtois and S. Gallot. Minimal entropy and Mostow's rigidity theorems. *Ergod. Th. & Dynam. Sys.* **16**(4) (1996), 623–649. MR1406425 (97e:58177)
7. N. Fenichel. Asymptotic stability with rate conditions. *Indiana University Math. J.* **23** (1974), 1109–1137; **26**(1) (1977), 81–93. MR0426056 (54 #14002)
8. L. W. Green. The generalized geodesic flow. *Duke Math. J.* **41** (1974), 115–126. MR0370659 (51 #6885)
9. M. Gerber and V. Niţică. Hölder exponents of horocycle foliations on surfaces. *Ergod. Th. & Dynam. Sys.* to appear MR1721619 (2000h:37042)
10. M. Grayson, C. Pugh and M. Shub. Stably ergodic diffeomorphisms. *Ann. Math.* **140**(2) (1994), 295–329. MR1298715 (95g:58128)
11. B. Hasselblatt. Regularity of the Anosov splitting and of horospheric foliations. *Ergod. Th. & Dynam. Sys.* **14**(4) (1994), 645–666. MR1304137 (95j:58130)
12. B. Hasselblatt. Horospheric foliations and relative pinching. *J. Diff. Geom.* **39**(1) (1994), 57–63. MR1258914 (95c:58137)
13. B. Hasselblatt. Regularity of the Anosov splitting II. *Ergod. Th. & Dynam. Sys.* **17**(1) (1997), 169–172. MR1440773 (98d:58135)
14. M. Hirsch, C. Pugh and M. Shub. *Invariant Manifolds (Lecture Notes in Mathematics, 583)*. Springer, 1977. MR0501173 (58 #18595)
15. S. Hurder and A. Katok. Differentiability, rigidity, and Godbillon-Vey classes for Anosov flows. *Publ. IHES* **72** (1990), 5–61. MR1087392 (92b:58179)
16. J.-L. Journé. A regularity lemma for functions of several variables. *Revista Mat. Iberoamericana* **4** (1988), 187–193. MR1028737 (91j:58123)
17. M. Kanai. Geodesic flows of negatively curved manifolds with smooth stable and unstable foliations. *Ergod. Th. & Dynam. Sys.* **8**(2) (1988), 215–239. MR0951270 (89k:58230)
18. A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995. MR1326374 (96c:58055)
19. F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms, Part I: characteri-

zation of measures satisfying Pesin's entropy formula. *Ann. Math.* **122**(3) (1985), 509–539. MR0819556 (87i:58101a)
20. R. de la Llave, J. M. Marco and R. Moriyon. Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation. *Ann. Math.* **123**(3) (1986), 537–611. MR0840722 (88h:58091)
21. C. Pugh and M. Shub. Stably ergodic dynamical systems and partial hyperbolicity. *J. Complexity* **13**(1) (1997), 125–179. MR1449765 (98e:58110)
22. C. Pugh, M. Shub and A. Wilkinson. Hölder foliations. *Duke Math. J.* **86**(3) (1997), 517–546. MR1432307 (97m:58155)
23. J. Schmeling and R. Siegmund-Schultze. Hölder continuity of the holonomy maps for hyperbolic basic sets. I. *Ergodic Theory and Related Topics, III, Güstrow, 1990 (Lecture Notes in Mathematics, 1514)*. Springer, Berlin, 1992, pp. 174–191. MR1179182 (93j:58104)
24. A. Wilkinson. Stable ergodicity of the time-one map of a geodesic flow. *Ergod. Th. & Dynam. Sys.* **18**(6) (1998), 1545–1587. MR1658611 (99m:58129)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1465582 (98g:58140) 58F18 (58F15)
Hasselblatt, Boris (1-TUFT); **Wilkinson, Amie** (1-NW)
Prevalence of non-Lipschitz Anosov foliations. (English summary)
Electron. Res. Announc. Amer. Math. Soc. **3** (1997), 93–98 (*electronic*).

The authors give sharp regularity results for invariant subbundles of Anosov dynamical systems. They exhibit open subsets in the C^1 topology of symplectic Anosov diffeomorphisms and flows such that the transverse Hölder regularity of invariant foliations is low almost everywhere in the manifold (i.e., given any $\alpha \in (0,1)$ the transverse holonomy is of class C^α only in a subset of measure zero). In particular, the authors give open dense subsets of Anosov systems where the analogous regularity results by Pugh, Shub and Wilkinson are optimal.

Reviewed by *Rafael Oswaldo Ruggiero*

References

1. D. V. Anosov, *Geodesic flows on closed Riemann manifolds with negative curvature*, Proc. Steklov Inst. **90** (1967). MR0224110 (36 #7157)
2. D. V. Anosov, *Tangential fields of transversal foliations in "U-systems"*, Mat. Zametki **2** (1967), no. 5, 818–823. MR0242190 (39 #3523)
3. G'erard Besson, Gilles Courtois, Sylvestre Gallot, *Minimal entropy and Mostow's rigid- ity*

theorems, Ergodic Theory and Dynamical Systems **16** (1996), no. 4, 623–649. MR1406425 (97e:58177)

4. Neil Fenichel, *Asymptotic stability with rate conditions*, Indiana University Math. Journal **23** (1974), 1109–1137; **26** (1977), no. 1, 81–93. MR **49:**4036, MR0426056 (54 #14002)
5. Matthew Grayson, Charles Pugh, Michael Shub, *Stably ergodic diffeomorphisms*, Annals of Mathematics (2) **140** (1994), no. 2, 295–329. MR1298715 (95g:58128)
6. Leon W. Green, *The generalized geodesic flow*, Duke Mathematical Journal **41** (1974), 115–126. MR0370659 (51 #6885)
7. Boris Hasselblatt, *Regularity of the Anosov splitting and of horospheric foliations*, Ergodic Theory and Dynamical Systems, **14** (1994), no. 4, 645–666. MR1304137 (95j:58130)
8. Boris Hasselblatt, *Horospheric foliations and relative pinching*, Journal of Differential Geometry **39** (1994), no. 1, 57–63. MR1258914 (95c:58137)
9. Boris Hasselblatt, *Periodic bunching and invariant foliations*, Mathematical Research Let- ters **1** (1994), no. 5, 597–600. MR1295553 (95h:58097)
10. Boris Hasselblatt, *Regularity of the Anosov splitting II*, Ergodic Theory and Dynamical Systems, **17** (1997), 169–172. CMP 97:10 MR1440773 (98d:58135)
11. Morris Hirsch, Charles Pugh, Michael Shub, *Invariant manifolds*, Lecture Notes in Mathematics **583**, Springer-Verlag, 1977. MR **58:**18595
12. Steven Hurder, Anatole Katok, *Differentiability, rigidity, and Godbillon–Vey classes for Anosov flows*, Publications IHES **72** (1990), 5–61. MR1087392 (92b:58179)
13. Anatole Katok, Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995. MR1326374 (96c:58055)
14. Rafael de la Llave, Roberto Moriyon, *Invariants for smooth conjugacy of hyperbolic dynamical systems. IV*, Communications in Mathematical Physics **116** (1988), no. 2, 185– 192. MR0939045 (90h:58064)
15. Sheldon Newhouse, *On codimension one Anosov diffeomorphisms*, American Journal of Mathematics **92** (1970), 761–770. MR0277004 (43 #2741)
16. Yakov Pesin, *On the existence of invariant fiberings for a diffeomorphism of a smooth manifold*, Math. USSR Sbornik **20** (1973), no. 2, 213–222. MR **49:**8049
17. Charles Pugh, Michael Shub, *Stably ergodic dynamical systems and partial hyperbolicity*, Journal of Complexity, *to appear* cf. MR 98e:58110
18. Charles Pugh, Michael Shub, Amie Wilkinson, *H"older foliations*, Duke Mathematical Journal, **86** (1997), no. 3, 517–546. MR1432307 (97m:58155)
19. J"org Schmeling, Rainer Siegmund-Schulze, *H"older continuity of the holonomy maps for hyperbolic basic sets, I*, Ergodic theory and related topics, III (G"ustrow, 1990), pp. 174– 191, Springer lecture notes in mathematics 1514, Springer, Berlin, 1992. MR1179182 (93j:58104)
20. Amie Wilkinson, *Stable ergodicity of the time-one map of a geodesic flow*, Ergodic Theory and Dynamical Systems, to appear cf. MR 99m:58129

Note: The reference list for this item has been removed for technical reasons.

MR1440773 (98d:58135) 58F15 (58F17)
Hasselblatt, Boris (1-TUFT)
Regularity of the Anosov splitting. II. (English summary)
Ergodic Theory Dynam. Systems **17** (1997), *no. 1,* 169–172.

The paper under review extends the author's previous result [Part I, Ergodic Theory Dynam. Systems **14** (1994), no. 4, 645–666; MR1304137 (95j:58130)] on the regularity of stable and unstable manifolds to the case of an integral bunching constant. A flow φ_t on a compact Riemannian manifold M is Anosov if the tangent bundle can be decomposed as $T = E^{su} \oplus E^{ss} \oplus E^{\varphi}$, where E^{φ} is the span of the flow associated to φ and there are $C, \varepsilon > 0$ such that for all $p \in M$ there are $\mu_f < \mu_s < 1 - \varepsilon$ and $1 + \varepsilon < \nu_s < \nu_f$ so that for $v \in E^{ss}(p)$ and $u \in E^{su}(p)$ and $t > 0$ we have $C^{-1}\mu_f^t||v|| \leq ||D\varphi_t(v)|| \leq C\mu_s^t||v||$ and $C^{-1}\nu_f^{-t}||u|| \leq ||D\varphi_{-t}(u)|| \leq C\nu_s^{-t}||u||$. The unstable bunching constant is $B^u(\varphi) = \inf(\log\mu_s - \log\nu_s)/\log\mu_f$ over $p \in M$. The distributions $E^u = E^{su} \oplus E^{\varphi}$ and $E^s = E^{ss} \oplus E^{\varphi}$ provide the Anosov splitting of T and their regularity is measured via their representation in smooth local coordinates. Set $\alpha = B^u(\varphi)$. The main result in this article is that $E^u \in C^{\alpha}$ if $\alpha \notin \mathbf{N}$ and $E^u \in C^{\alpha, x|\log x|}$ if $\alpha \in \mathbf{N}$. A function f is β-Hölder continuous, $f \in C^{\beta}$, when f has $[\beta]$ continuous derivatives and $f^{([\beta])}$ is Hölder continuous with modulus $\beta - [\beta]$. One writes $f \in C^{\beta, x|\log x|}$ when $g = f^{([\beta])}$ has modulus of continuity

$$|g(y) - g(x)| \leq C|x - y| \cdot |\log|x - y||$$

(for x and y sufficiently close).

Reviewed by *Edoh Amiran*

References

1. U. Hamenstädt. Compact manifolds with $\frac{1}{4}$-pinched negative curvature. *Global Differential Geometry and Global Analysis (Lecture Notes in Mathematics 1481)* Proceedings of a conference held in Berlin, 15–20 June, 1990. Eds D. Ferus, U. Pinkall, U. Simon and B. Wegner. Springer, Berlin, Heidelberg, New York, 1991, pp. 73–78. MR1178520 (93i:53036)
2. B. Hasselblatt. Regularity of the Anosov splitting and of horospheric foliations. *Ergod. Th. & Dynam. Sys.* **14** (1994), 645–666. MR1304137 (95j:58130)
3. B. Hasselblatt. Anosov obstructions in higher dimension. *Int. J. Math.* **4** (1993), 395–407. MR1228581 (95d:58102)
4. B. Hasselblatt. Horospheric foliations and relative pinching. *J. Diff. Geometry*. **39** (1994), 57–63. MR1258914 (95c:58137)
5. B. Hasselblatt. Periodic bunching and invariant foliations. *Mathematical Research Letters*. **1** (1994), 597–600. MR1295553 (95h:58097)
6. S. Hurder and A. Katok. Differentiability, rigidity and Godbillon-Vey classes for Anosov flows. *Publ. I.H.E.S.* **72** (1990), 5–61. MR1087392 (92b:58179)

7. A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, New York, Cambridge, 1995. MR1326374 (96c:58055)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1326374 (96c:58055) 58Fxx (34Cxx 34Dxx 58-01 58F11 58F15)
Katok, Anatole (1-PAS); **Hasselblatt, Boris** (1-TUFT)
★**Introduction to the modern theory of dynamical systems. (English summary)**
With a supplementary chapter by Katok and Leonardo Mendoza.
Encyclopedia of Mathematics and its Applications, 54.
Cambridge University Press, Cambridge, 1995. xviii+802 *pp*. $69.95. *ISBN* 0-521-34187-6

Most introductory texts in dynamical systems concern somewhat limited systems, such as homeomorphisms of the interval, or only particular techniques, such as symbolic dynamics or simulation of bifurcation. The book under review is an introduction to differentiable dynamical systems and all that is connected to their analysis. Thus it must include thorough treatments of topological dynamics, symbolic dynamics, and ergodic theory.

In order to begin a comprehensive exposition without sacrificing motivation, the authors use examples interlaced with definitions and propositions in the first chapter. Later chapters are organized by topic, providing easier reference and some independence among chapters.

This book emphasizes topological, measure-theoretic, and number-theoretic invariants associated to dynamical systems, and methods for deciding a system's asymptotic behavior, including many local-to-global results.

The motivation is evident throughout the book. For example, the third chapter, on principal classes of asymptotic topological invariants, begins: "In this chapter we will embark upon the task of systematically identifying important specific phenomena associated with the asymptotic behavior of smooth dynamical systems. We will build upon the results of our survey of specific examples in Chapter 1 as well as on the insights gained from the general structural approach outlined and illustrated in Chapter 2."

The (over 650 pages of) text is supplemented with a discussion of nonuniformly hyperbolic systems by Katok and L. Mendoza, with an appendix providing background material, with historical notes, and with hints and answers to exercises.

The audience for this book consists of graduate students or researchers in mathematics or related fields who have the "first year" background in measure theory, functional analysis, topology (with some differential geometry), and algebra, or are willing to learn it quickly from the appendix. The book is a pleasure to read.

Reviewed by *Edoh Amiran*

MR1304137 (95j:58130) 58F15 (58F17 58F18)
Hasselblatt, Boris (1-TUFT)
Regularity of the Anosov splitting and of horospheric foliations. (English summary)
Ergodic Theory Dynam. Systems **14** (1994), *no. 4,* 645–666.

The author defines a flow on a smooth compact Riemannian manifold to be Anosov, using bunching conditions on the strong stable and strong unstable distributions at a point. (Localization follows the ideas of S. E. Hurder and A. Katok [Inst. Hautes Études Sci. Publ. Math. No. 72 (1990), 5–61 (1991); MR1087392 (92b:58179)].) Relating the regularity of the stable and unstable distributions to the bunching parameter, it is shown that under certain conditions the Anosov splitting is $C^{2-\varepsilon}$, as conjectured in the late 1970s by Hirsch, Pugh and Shub, and others.

The main aim of the paper is to show that this regularity is optimal. The author shows that the regularity of the Anosov splitting in dimension 4 or greater and of the horosphere foliation of negatively curved manifolds of dimension 3 or greater which have a point at which the injectivity radius exceeds $\log 2/\sqrt{K}$ is generically no better than the above. Several further results are obtained for special cases.

The principal technique developed in this paper is to examine geometric consequences of bunching. The unstable distribution is represented as the graph of a linear map and it is shown that high regularity of the "portion" of the graphed map corresponding to the slowest expansion and contraction under the flow leads to a "horizontality" condition which is generically violated.

Reviewed by *Edoh Amiran*

MR1295553 (95h:58097) 58F15 (58F17 58F18)
Hasselblatt, Boris (1-TUFT)
Periodic bunching and invariant foliations. (English summary)
Math. Res. Lett. **1** (1994), *no. 5,* 597–600.

Let $\varphi_t\colon M\to M$ denote a transitive Anosov flow on a compact manifold M. A periodic point p of period τ is said to be α-u-bunched if the absolute values of the eigenvalues of $d\varphi_\tau(p)$ restricted to the stable bundle are contained in the interval $[q_3,q_1]$ and the absolute values of the eigenvalues of $d\varphi_\tau(p)$ restricted to the unstable bundle are bounded by q_2, where $\log q_1+\log q_2-\alpha\log q_3\le 0$ (q_1, q_2 and q_3 can vary with the periodic point p). The period orbits are said to be α-bunched if all the periodic points are α-u-bunched under both φ_t and φ_{-t}.

The author shows, among other things, that if the periodic orbits are α-bunched, then the Anosov splitting is of class $C^{\alpha-\varepsilon}$. These results complete and are based on previous results of the author [Ergodic Theory Dynam. Systems **14** (1994), no. 4, 645–666] and U. Hamenstädt [Ergodic Theory Dynam. Systems **14** (1994), no. 2, 299–304; MR1279472 (95c:58132)].

Reviewed by *Gabriel P. Paternain*

MR1258914 (95c:58137) 58F17 (53C20 58F15 58F18)
Hasselblatt, Boris (1-TUFT)
Horospheric foliations and relative pinching. (English summary)
J. Differential Geom. **39** (1994), *no. 1,* 57–63.

M. W. Hirsch and C. C. Pugh [J. Differential Geom. **10** (1975), 225–238; MR0368077 (51 #4319)] showed that absolutely $\frac{1}{4}$-pinched, negatively curved Riemannian manifolds have C^1 horospheric foliations. This result was later improved to show that a-pinched, negatively curved manifolds have $C^{2\sqrt{a}}$ horospheric foliations, $a\in(0,1)$. Hirsch and Pugh raised the question of whether the same is true for relative pinching, i.e., for negatively curved manifolds satisfying a pointwise pinching condition. This article gives a partial answer by establishing that relatively a-pinching implies C^{2a} horospheric foliations, $a\in(0,1)$. The result is proved by relating relative a-pinching to the notion of bunching for Anosov flows, and builds on earlier work of the author [B. Hasselblatt, Ergodic Theory Dynamical Systems **14** (1994), no. 4, 645–666].

Reviewed by *Garrett Stuck*

MR1228581 (95d:58102) 58F15 (58F17 58F18)
Hasselblatt, Boris (1-TUFT)
Anosov obstructions in higher dimension. (English summary)
Internat. J. Math. **4** (1993), *no. 3,* 395–407.

From the text: "Let M be a compact Riemannian manifold. $f \in \mathrm{Diff}^\infty(M)$ is said to be Anosov with Anosov splitting $TM = E^u \oplus E^s$ if there exist $\lambda < 1$ and C, such that for all $p \in M$, $n \in \mathbf{N}$, $\|Df^n(v)\| \le C\lambda^n\|v\|$ $(v \in E^s(p))$ and $\|Df^{-n}(u)\| \le C\lambda^{-n}\|u\|$ $(u \in E^u(p))$. The regularity of the Anosov splitting has been studied in several works—for example earlier by Anosov, Hirsch and Pugh and more recently by Hurder, Katok and the author. A paper by S. E. Hurder and A. Katok [Inst. Hautes Études Sci. Publ. Math. No. 72 (1990), 5–61 (1991); MR1087392 (92b:58179)] contains a complete analysis for two-dimensional area-preserving Anosov diffeomorphisms using D. V. Anosov's observation [*Geodesic flows on closed Riemannian manifolds with negative curvature*, English translation, Amer. Math. Soc., Providence, RI, 1969; MR0242194 (39 #3527)] that at every periodic point there is an obstruction to the Anosov splitting being C^2 by showing that it arises from a cocycle. By work of B. Hasselblatt ["Regularity of the Anosov splitting and of horospheric Foliations", Ergodic Theory Dynamical Systems, to appear] not all aspects of the work of Hurder and Katok [op. cit.] generalize to higher dimension: For symplectic systems generically the Anosov splitting is less regular than in the two-dimensional case. But the Anosov cocycle does have a counterpart. The results apply to flows, e.g., geodesic flows (Propositions 12 and 19 assert that symmetric spaces can be perturbed so that the horospheric folations are nonsmooth). These applications only depend on Theorem 1 below, not the more detailed study of the obstruction that follows. We obtain the C^2-obstruction at periodic points in the first theorem. Then we give properties analogous to the two-dimensional case, in particular we observe that sometimes the obstruction arises from the cohomology class of a cocycle defined at every point. We use the notions of fixed and periodic points interchangeably to avoid iterating the diffeomorphism. Two appendices study adapted local coordinates and C^1-foliations."

AMERICAN MATHEMATICAL SOCIETY
MathSciNet Mathematical Reviews on the Web
Article

Citations
From References: 1
From Reviews: 0

MR1101985 (92i:58138) 58F15 (58F18)

Hasselblatt, Boris (1-TUFT)

Bootstrapping regularity of the Anosov splitting.

Proc. Amer. Math. Soc. **115** (1992), *no. 3,* 817–819.

Let f be a C^∞ Anosov diffeomorphism of a compact Riemannian manifold. The main result is that if the Anosov splitting $TM = E^u \oplus E^s$ of f is C^n, where n depends only on the constants entering the definition of the splitting, then it is C^∞. The same holds for Anosov flows. Some corollaries are derived.

Reviewed by *Valery Covachev*

MR1016664 (91b:58192) 58F15 (58F11)

Hasselblatt, Boris (1-TUFT)

A new construction of the Margulis measure for Anosov flows.

Ergodic Theory Dynam. Systems **9** (1989), *no. 3,* 465–468.

U. Hamenstädt's description [same journal **9** (1989), no. 3, 455–464; see the preceding review] of the Margulis measure for geodesic flows on compact manifolds of negative curvature is generalised to Anosov flows. Let φ^t be an Anosov flow on M, let $W^{\rm u}(z)$ denote the unstable manifold at z, and let $TM = E^{\rm u} \oplus E^{\rm s} \oplus E^0$ be the decomposition of the tangent bundle, where $E^{\rm u}$ is the unstable bundle, E^0 is tangent to the flow and $E^{\rm s}$ is the stable bundle. The new description of the Margulis measure, much as in Hamenstädt's paper, needs a Riemannian metric on the $W^{\rm u}(z)$, coming from a Riemannian metric on M, which can, in fact, be any Lyapunov metric: that is, for some $a > 0$, $\|D\varphi^t u\| \le e^{at}\|u\|$ for $t \le 0$, $u \in E^u$, $\|D\varphi^t v\| \le e^{-at}\|v\|$ for $t \ge 0$, $v \in E^s$. After that, the description of the measure is exactly as in Hamenstädt's paper. The proof that the new description does give the original Margulis measure seems to be a bit different, though the crucial point—the identity $\eta \circ \varphi^t = e^t \cdot \eta$ (the notation is the same as in Hamenstädt's paper)—is the same.

Reviewed by *M. Rees*