Solutions for Homework Assignment 4
Math/Comp 128

1. Since \( P \) is an orthogonal projector, we know that \( P = P^2 \) and \( P = P^* \). We want to show that \( I - 2P \) is unitary. It is enough to show that \( (I - 2P)^*(I - 2P) = I \). We have that

\[
\]

2. Given a matrix \( A \), the orthogonal projector \( P \) onto range(\( A \)) is given by \( P = A(A^*A)^{-1}A^* \). Using this last formula, we have that

\[
P = \begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix}.
\]

The image under \( P \) is given by

\[
\begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & 1 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix} \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = \begin{bmatrix}
\frac{2}{2} \\
2 \\
2
\end{bmatrix}.
\]

3. The reduced \( QR \) factorization of \( A \) is given by

\[
A = \begin{bmatrix}
\sqrt{2}/2 & 0 \\
0 & 1 \\
\sqrt{2}/2 & 0
\end{bmatrix} \begin{bmatrix}
\sqrt{2} & 0 \\
0 & 1
\end{bmatrix}.
\]

A full \( QR \) factorization of \( A \) is given by

\[
A = \begin{bmatrix}
\sqrt{2}/2 & 0 & \sqrt{2}/2 \\
0 & 1 & 0 \\
\sqrt{2}/2 & 0 & -\sqrt{2}/2
\end{bmatrix} \begin{bmatrix}
\sqrt{2} & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}.
\]

3. a) Let \( A = QR \) be the reduced \( QR \) factorization of \( A \). If we write \( A^*A \) in terms of this decomposition, we get

\[
A^*A = R^*Q^*QR = R^*R.
\]

Since \( A \) is full rank, the entries on the diagonal of \( R \) are all different from zero. So \( R \) and \( R^* \) are invertible. Then, \( A^*A \) is invertible.

3. b)
\[
(A^*A)^{-1} = R^{-1}R^{-*} \\
(A^*A)^{-1}A^* = R^{-1}Q^*
\]
\[A(A^*A)^{-1} = QR^*\]
\[A(A^*A)^{-1}A^* = QQ^*\]

3. c)
\[x = (A^*A)^{-1}A^*b = R^{-1}Q^*b\]
\[r = b - Ax = b - (QR)(R^{-1}Q^*b) = b - QQ^*b = (I - QQ^*)b\]

Solutions for Math 250NLA

1. Consider the SVD of \(A = U\Sigma V^*\). Then, we have that the SVD of \(A^*A = V\Sigma^2 V^*\). We know that \(A\) is full rank \(\iff\) \(\Sigma\) has all its diagonal entries different from zero \(\iff\) \(\Sigma^2\) has all its diagonal entries different from zero \(\iff\) \(A^*A\) is nonsingular.

2. See solution 2 of Math/Comp 128

3. See solution 3 of Math/Comp 128

4. Using that \(r_{ij} = q_i^*a_j\) for \(i \neq j\), and that \(\langle a_n, a_m \rangle = 0\) for \(n + m\) odd, we can show by induction that \(r_{nm} = 0\) for \(n + m\) odd.

5. Using the result of Problem 1., we know that \(A\) is full rank \(\iff\) \(A^*A\) is nonsingular. Therefore, in order to prove the statement of the problem, it is enough to show that \(A^*A\) is nonsingular \(\iff\) \(R\) has its diagonal entries different from zero.

   We know that \(A^*A = R^*R\). If \(R\) has its diagonal entries different from zero, then it is invertible and so is \(R^*\). Then, \(A^*A\) is nonsingular.

   Now, assume that \(A^*A\) is nonsingular. Then, \(\text{Null}(A^*A) = \{0\}\). Suppose that \(R\) has an element in its diagonal equal to zero. Since \(R\) is upper triangular, then it is singular. That is, there exists a vector \(c \neq 0\) such that \(Rc = 0\). Moreover, \(R^*Rc = 0\). Thus, we found a vector \(c \neq 0\) such that \(A^*Ac = 0\). This contradicts the fact that \(\text{Null}(A^*A) = \{0\}\). We conclude that \(R\) has all its diagonal entries different from zero.