(65 points) In problems 1–7

a. State the order of the differential equation.

b. If the differential equation is of first order, state whether it is separable or not.

c. If the differential equation is separable, find the general solution and skip parts d)–f).

d. State whether the differential equation is linear. If it is not, skip parts e)–f).

e. If the differential equation is linear find the largest interval containing 1 on which the differential equation is normal; state whether the differential equation is homogeneous or nonhomogeneous, and whether it has constant coefficients or not.

f. If the differential equation is linear, find the general solution.

1. \( t^2 x' = x^2 \).

   **Answer:** a. 1, b. yes, c. \( x = \frac{1}{(1/t) + C} \).

2. \( x' + x^2 + t = 0 \).

   **Answer:** a. 1, b. no, d. no.

3. \( x' x + t = 0 \).

   **Answer:** a. 1, b. yes, c. \( x = \pm \sqrt{C - t^2} \).

4. \( tx' - 3x = t^3 + 2t \).

   **Answer:** a. 1, b. no, d. yes, e. \((0, \infty)\), nonhomogeneous, nonconstant coefficients, f. \( t^3 \ln t - t + ct^3 \).

5. \( 2x'' - 5x = 0 \).

   **Answer:** a. 2, d. yes, e. \( \mathbb{R} \), homogeneous, constant coefficients, f. \( c_1 e^{\sqrt{5/2}t} + c_2 e^{-\sqrt{5/2}t} \).

6. \( 9x'' - 12x' + 4x = 0 \).

   **Answer:** a. 2, d. yes, e. \( \mathbb{R} \), homogeneous, constant coefficients, f. \( c_1 e^{2t/3} + c_2 e^{2t/3} \).

7. \( (D^2 + 1)^2 x = 0 \).

   **Answer:** a. 6, d. yes, e. \( \mathbb{R} \), homogeneous, constant coefficients, f. \( c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t + c_5 t^2 \cos t + c_6 t^2 \sin t \).
8. (7 points) Sketch the phase portrait of \( x' = \frac{(x - 1)(x + 2)^2}{x^2 + 3} \), find all equilibria, determine their stability, and classify them as attractors or repellers, or neither of these.

**Answer:** The equilibria are at 1 and \(-2\); 1 is a repeller, hence unstable, \(-2\) is neither an attractor nor a repeller, and it is unstable.

9. (8 points) Find all solutions of the form \( e^{\lambda t} \) or \( t^\alpha \) of \( tx'' + 2x' = 0 \) and decide whether those solutions form a complete collection of solutions.

**Solution:** Plugging in \( t^\alpha \) gives \( \alpha(\alpha - 1)t^{\alpha - 1} + 2\alpha t^{\alpha - 1} = 0 \) for all \( t \), hence \( \alpha = 0 \) or \(-1\), giving solutions 1 and \( 1/t \). Since the Wronskian of these is \( \det \begin{pmatrix} 1 & 1/t \\ 0 & -1/t^2 \end{pmatrix} = -1/t^2 \neq 0 \), these determine a complete collection of solutions (and therefore there is no need to look for solutions of the form \( e^{\lambda t} \)).

10. (10 points) Determine whether the system

\[
\begin{align*}
    x - y + 3z &= a \\
    x + y - 3z &= b \\
    2x - z &= c
\end{align*}
\]

has solutions for all values on the right-hand side.

**Solution:** Since \( \det \begin{pmatrix} 1 & -1 & 3 \\ 1 & 1 & -3 \\ 2 & 0 & -3 \end{pmatrix} = 2 \cdot 0 - 3 \cdot 2 = -6 \neq 0 \), it does.

11. (10 points) Given that \( a, b, c \) are distinct constants decide whether the functions \( e^{\alpha t}, e^{bt}, \) and \( e^{ct} \) are linearly independent.

**Solution:** The Wronskian at \( t = 0 \) is

\[
\det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2 - a^2 & c^2 - a^2 \end{pmatrix} = (b-a)(c^2 - a^2) - (c-a)(b^2 - a^2),
\]

and \( (b-a)(c^2 - a^2) - (c-a)(b^2 - a^2) = (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a) = (b-a)(c-a)(c-b) \neq 0 \) since \( a, b, c \) are distinct constants. Therefore, the functions \( e^{\alpha t}, e^{bt}, \) and \( e^{ct} \) are linearly independent.