ZYGMUND STRONG FOLIATIONS

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ABSTRACT. We show that for a volume-preserving Anosov flow on a 3-manifold the strong stable and unstable foliations are Zygmund-regular. We also exhibit an obstruction to higher regularity, which admits a direct geometric interpretation. Vanishing of this obstruction implies high smoothness of the joint strong subbundle and that the flow is either a suspension or a contact flow.

1. INTRODUCTION

In their study of volume-preserving Anosov flows on 3-manifolds Hurder and Katok showed that the weak-stable and weak-unstable foliations are $C^{1+\text{Zygmund}}$, and that there is an obstruction to higher regularity whose vanishing implies smooth foliations. We show that the strong stable and unstable foliations are Zygmund-regular and that there is an obstruction to higher regularity, which (unlike that for higher regularity of the weak foliations) admits a direct geometric interpretation, and whose vanishing implies smooth joint strong subbundle $E^u \oplus E^s$.

Definition 1.1 ([KH]). Let M be a manifold, $\varphi\colon \mathbb{R} \times M \to M$ a smooth flow. Then φ is said to be an Anosov flow if the tangent bundle TM splits as $TM = E^\varphi \oplus E^u \oplus E^s$ in such a way that there are constants $C > 0 < \lambda < 1 < \eta$ such that $E^\varphi(x) = \mathbb{R}\dot{\varphi}(x) \neq \{0\}$ for all $x \in M$ and for $t > 0$ we have

$$\|D\varphi^{-t}\restriction_{E^u}\| \leq C\eta^{-t} \text{ and } \|Df^t\restriction_{E^s}\| \leq C\lambda^t.$$

The subbundles are then invariant and (Hölder-) continuous and have smooth integral manifolds W^u and W^s that are coherent in that $q \in W^u(p) \implies W^u(q) = W^u(p)$. W^u and W^s define laminations (continuous foliations with smooth leaves). The aim of this paper is to show that if φ preserves volume and all three subbundles are one-dimensional then they are Zygmund-regular [Z, Section II.3, (3·1)] and that there is an obstruction to higher regularity that can be described in geometric terms.

Definition 1.2. A function f between metric spaces is said to be *Hölder* continuous if there is an $H > 0$, called the Hölder exponent, such that $d(f(x), f(y)) \leq \text{const.}\, d(x, y)^H$ whenever $d(x, y)$ is sufficiently small. We specify the constant by saying that a function is H-Hölder. A continuous function $f\colon U \to \mathbb{R}$ on an open set $U \subset \mathbb{R}$ is said to be Zygmund-regular if there is $K > 0$ such that $|f(x + h) + f(x - h) - 2f(x)| \leq K|h|$ for all $x \in U$ and sufficiently small h. To specify a value of K we may refer to a function as being K-Zygmund. The function is said to be "little Zygmund" if $|f(x + h) + f(x - h) - 2f(x)| = o(|h|)$.

Zygmund regularity implies modulus of continuity $O(|x \log |x||)$ and hence H-Hölder continuity for all $H < 1$ [Z, Theorem (3·4)]. It follows from Lipschitz continuity and hence from differentiability. Being "little Zygmund" implies having modulus of continuity $o(|x \log |x||)$.

The background to this investigation is the paper by Hurder and Katok [HK], which, in the same context, proves a similarly sharp regularity result for the weak-unstable subbundle $E^u \oplus E^\varphi$: It is differentiable with Zygmund-regular derivative, and there is an obstruction to higher regularity of the derivative. Indeed, this obstruction vanishes only if the Anosov flow is smoothly conjugate to an algebraic one. The cocycle obstruction described by *K*atok and Hurder was first observed by *A*nosov and is the first nonlinear coefficient in the *M*oser normal form. Therefore one might call it the *KAM*-cocycle.

The regularity of the unstable subbundle E^u is usually substantially lower than that of the weak-unstable subbundle. The exception are geodesic flows, where the strong unstable subbundle is obtained from the weak-unstable subbundle by intersecting with the kernel of the invariant contact form. This has the effect that the strong-unstable and weak-unstable subbundles have the same regularity. However, time changes affect the regularity of the strong-unstable subbundle, and this is what typically keeps its regularity below C^1. We present a variant of the *KAM*-cocycle, the longitudinal KAM-cocycle, that is the obstruction to differentiablility.

Theorem 1.3. *Let M be a 3-manifold, $k \geq 2$, $\varphi\colon \mathbb{R} \times M \to M$ a C^k volume-preserving Anosov flow. Then $E^u \oplus E^s$ is Zygmund-regular, and there is an obstruction to higher regularity that can be described geometrically as the curvature of the image of a transversal under a return map. This obstruction defines the cohomology class of a cocycle (the longitudinal KAM-cocycle), and the following are equivalent:*

1. *$E^u \oplus E^s$ is "little Zygmund" (see Definition 1.2).*
2. *The longitudinal KAM-cocycle is a coboundary.*
3. *$E^u \oplus E^s$ is Lipschitz.*
4. *$E^u \oplus E^s \in C^{k-1}$.*
5. *φ is a suspension or contact flow.*

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2. Zygmund regularity

In this section we prove Zygmund regularity of $E^u \oplus E^s$. We use the Hadamard graph transform method [H, KH]. It was developed in order to prove the existence of invariant manifolds, and here we examine it with a view to regularity of the subbundles. We apply it to the one-form whose kernel is $E^u \oplus E^s$.

The graph transform acts on subbundles by $\varphi^t_* E(p) := D\varphi^t(E(\varphi^{-t}(p)))$, *i.e.*, the subbundle E is acted on by the differential of φ^t. If one considers the space of continuous subbundles moderately close to E^u with the distance defined by (the supremum of) pointwise angles, then the graph transform is a contracting map and hence has a unique fixed point, E^u. In order to prove regularity of E^u it therefore suffices consider the same transformation in a complete subspace of distributions of the desired regularity, and to prove that the orbit of some intial distribution remains in that space. This proves that the fixed point E^u is in that space as well.

In fact, there is a little less to show that it would appear: The subbundle E^u is smooth in the flow direction by invariance, and it is C^{k-1} along W^u because W^u has C^k leaves. Therefore it suffices to show that E^u is Zygmund-regular along W^s.

Take $T > 0$, fixed for now. After possibly rescaling time we will from now assume that φ^T contracts stable manifolds and expands unstable manifolds. We consider the graph transform φ^T_* henceforth.

Lemma 2.1. *There exist local coordinates adapted to the invariant laminations,* i.e., *coordinate systems* $\Psi\colon M \times (-\epsilon, \epsilon)^3 \to M$ *such that* $\Psi_p := \Psi(p, \cdot)$ *satisfies*

1. Ψ_p *is* C^k *for every* $p \in M$.
2. Ψ_p *depends (Hölder-) continuously on* p.
3. Ψ_p *preserves volume for each* $p \in M$.
4. $\Psi_p(0) = p$.
5. $\Psi_p((-\epsilon, \epsilon) \times \{0\} \times \{0\}) = W^u_{loc}(p) \cap \Psi_p((-\epsilon, \epsilon)^3)$.
6. $\Psi_p^{-1}(\varphi^\delta(\Psi_p(u, t, s))) = (u, t + \delta, s)$ *for* $|\delta| < \epsilon$.
7. $\Psi_p(\{0\} \times \{0\} \times (-\epsilon, \epsilon)) = W^s_{loc}(p) \cap \Psi_p((-\epsilon, \epsilon)^3)$.

Such coordinates can be obtained easily [KH], and even more sophisticated adaptations are possible [HK, dL]. The basic ingredient is Moser's homotopy trick [KH], refined to impose conditions beyond volume normalization. It is natural to denote the coordinate variables by (u, t, s).

Since the subbundles are invariant under the flow, the coordinate representation of the flow preserves the axes of the local coordinate system as well as volume.

The differential of φ^T at points of the stable leaf (the third coordinate axis, or s-axis) therefore takes the following form:

$$D\varphi^T(0, 0, s) = \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ b_1 & 1 & 0 \\ b_2 & 0 & \alpha \end{pmatrix},$$

where $b_i(s) = O(s)$ and $|\alpha(s)| < 1$.

Proposition 2.2. *For each* $H \in (0, 1)$ *there is a* $K > 0$ *such that the graph transform preserves the space of subbundles that are* H*-Hölder along* E^s *with constant* K *in local coordinates.*

Thus the graph transform preserves Hölder continuity. More specifically, applying the graph transform to a subbundle that is H-Hölder with sufficiently large constant K in local coordinates gives a subbundle with the same property (for the same K and H). This holds for any $H < 1$. This implies immediately that E^u is H-Hölder for any $H < 1$, because it shows that the unique fixed point of the graph transform lies in the space of H-Hölder subbundles.

While this fact is not new, it may be illuminating to demonstrate it before proving Zygmund regularity.

Proof. In adapted coordinates a subbundle transverse to $E^\varphi\oplus E^s$ is represented by graphs of linear maps from E^u to $E^\varphi\oplus E^s$. Using the canonical representation of the tangent bundle of $\mathbb{R}^3$ we can write any subspace transverse to the ts-plane as the *image* of a linear map given by a column matrix $\begin{pmatrix}1\\ e\\ \bar e\end{pmatrix}$. Accordingly, the restriction of such a subbundle to stable leaves is given locally by matrices $\begin{pmatrix}1\\ e(s)\\ \bar e(s)\end{pmatrix}$. The advantage of this representation is that applying the derivative amounts to simple composition. The image (in the coordinates at $\varphi^T(p)$ of the subspace is the represented as the image of the linear map with matrix

$$\begin{pmatrix}\alpha^{-1} & 0 & 0\\ b_1 & 1 & 0\\ b_2 & 0 & \alpha\end{pmatrix}\begin{pmatrix}1\\ e\\ \bar e\end{pmatrix}=\begin{pmatrix}\alpha^{-1}\\ b_1+e\\ b_2+\bar e\alpha\end{pmatrix},$$

which is also the image of the linear map with matrix

$$\begin{pmatrix}1\\ \alpha(s)b_1(s)+\alpha(s)e(s)\\ \alpha(s)b_2(s)+\bar e(s)\alpha^2(s)\end{pmatrix}=:\begin{pmatrix}1\\ e(s_\varphi)\\ \bar e(s_\varphi)\end{pmatrix}, \tag{2.1}$$

where $\varphi^T(0,0,s)=:(0,0,s_\varphi)$ in local coordinates.

To prove Proposition 2.2 assume that $|e(s)|\le K|s|^H$ with uniform K and H. Then

$$\begin{aligned}|e(s_\varphi)|&=|\alpha(s)||b_1(s)+e(s)|\\ &\le|\alpha(s)|[O(s)+K|s|^H]\\ &=|\alpha(s)s|^H[K|\alpha|^{1-H}+|\alpha(s)|^{1-H}O(s^{1-H})]\\ &=|s_\varphi|^H(1+o(s))[K|\alpha|^{1-H}+|\alpha(s)|^{1-H}O(s^{1-H})]\le K|s_\varphi|^H\end{aligned}$$

for sufficiently large K. Likewise, assuming $|\bar e(s)|\le K|s|^H$ with uniform K and $H\in(0,1]$ gives

$$\begin{aligned}|\bar e(s_\varphi)|&=|\alpha(s)b_2(s)+\bar e(s)\alpha^2(s)|\\ &=|\alpha(s)s|^H[|\alpha(s)|^{1-H}O(|s|^{1-H})+K|\alpha(s)|^{2-H}]\\ &=|s_\varphi|^H(1+o(s))[|\alpha(s)|^{1-H}O(|s|^{1-H})+K|\alpha(s)|^{2-H}]\le K|s_\varphi|^H\end{aligned}$$

for sufficiently large K and sufficiently small s. $\square$

We return to this argument later.

To prove Zygmund regularity of the strong unstable subbundle we vary the strategy slightly. Instead of showing that the space of K-Zygmund subbundles is preserved for sufficiently large K, we show that for each K there is a $K'>0$ such that repeated application of the graph transform to a K-Zygmund subbundle always gives K'-Zygmund subbundles. Since the space of K'-Zygmund subbundles is closed, this proves that the unique fixed point E^u is K'-Zygmund.

Proposition 2.3. *For each $K > 0$ there is a $K' > 0$ such that all forward images under the graph transform of the space of subbundles transverse to $E^s \oplus E^\varphi$ that are K-Zygmund in local coordinates lie in the space of K'-Zygmund subbundles in local coordinates.*

Proof. Consider a subbundle that is represented in local coordinates as $\begin{pmatrix}1\\ e\\ \bar{e}\end{pmatrix}$ with $|e(s)+e(-s)| \leq K|s|$ and $|\bar{e}(s)+\bar{e}(-s)| \leq K|s|$ for all s in any local coordinate system.

Then $|e(s_\varphi)+e(-(s_\varphi))| \leq |e(s_\varphi)+e((-s)_\varphi)| + |e((-s)_\varphi)+e(-(s_\varphi))|$, where the last term is $O(y^{2H})$ for any $H < 1$ by Proposition 2.2 because $|s_\varphi + (-s)_\varphi| = \|\varphi(0,0,s)+\varphi(0,0,-s)\| = O(s^2)$ since φ is C^2.

The other term is estimated as follows:

$$\begin{aligned}
|e(s_\varphi)+e((-s)_\varphi)| &= |\alpha(s)b_1(s)+\alpha(-s)b_1(-s)+\alpha(s)e(s)+\alpha(-s)e(-s)| \\
&\leq |\alpha(s)||b_1(s)+b_1(-s)| + |b_1(-s)||\alpha(s)-\alpha(-s)| \\
&\quad + |\alpha(s)||e(s)+e(-s)| + |e(-s)||\alpha(s)-\alpha(-s)| \\
&\leq |\alpha(s)|O(s^2) + O(s)O(s) + |\alpha(s)|K|s| + O(s^H)O(s) \\
&\leq K(1+O(s^H))|\alpha(s)s| \\
&= K(1+O(s^H))(1+o(s))|s_\varphi| \\
&= K(1+\kappa(s))|s_\varphi|,
\end{aligned}$$

with $\kappa(s) = O(s^H)$ decreasing in K.

Note that $K' := K\prod_{i=0}^{\infty}(1+\kappa(s_{\varphi^i})) < \infty$ and all images of $\begin{pmatrix}1\\ e\\ \bar{e}\end{pmatrix}$ under the graph transform are K'-Zygmund in local coordinates. □

Corollary 2.4. *Let M be a 3-manifold, $\varphi\colon \mathbb{R}\times M \to M$ a C^k volume-preserving Anosov flow. Then $E^u \oplus E^s$ is Zygmund-regular.*

3. An obstruction to higher regularity

The regularity in Corollary 2.4 is sharp. It is easy to see that differentiability of the strong stable or unstable foliation cannot be expected.

Lemma 3.1. *There is an obstruction to differentiability of the strong unstable subbundle.*

Proof. Suppose p is a periodic point and take T to be its period. Differentiating (2.1) at 0 gives $e'(0)\alpha(0) = \alpha(0)b_1'(0)+\alpha(0)e'(0)$, and hence $K(p,T) := b_1'(0) = 0$. □

We sharpen this conclusion significantly in Proposition 3.4.

Note that $K(p,T)$ is naturally defined as a second order partial derivative of φ^T in adapted coordinates at any point $p \in M$. There is a natural geometric interpretation. Consider the transversals $\Delta := \psi_p((-\epsilon,\epsilon)\times\{0\}\times(-\epsilon,\epsilon))$ and $\Delta' := \psi_{\varphi^T(p)}((-\epsilon,\epsilon)\times$

$\{0\} \times (-\epsilon, \epsilon))$. Then $\Delta' \cap \varphi^T(\Delta)$ contains local strong stable and unstable manifolds of $\varphi^T(p)$, but the two transversals are not usually identical. As one sees from the coordinate represenatation of the flow, the obstruction gives the off-diagonal term in the Hessian of the map $(-\delta, \delta)^2 \to \mathbb{R}$ that gives the lengths of the orbit segments between Δ' and $\varphi^T(\Delta')$. This can be viewed as the "relative curvature" of image transversal versus original transversal. Lemma 4.1 shows that if the obstruction vanishes we can choose transversals such that the image transversals agree to third order.

Lemma 3.2. *K is an additive cocycle.*

Proof. We need to show that $K(p, T+S) = K(\varphi^T(p), S) + K(p, T)$ for all $p \in M$, $T, S \in \mathbb{R}$. Writing

$$D\varphi^T(0,0,s) = \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ b_1 & 1 & 0 \\ b_2 & 0 & \alpha \end{pmatrix}$$

at p and

$$D\varphi^S(0,0,s) = \begin{pmatrix} \bar{\alpha}^{-1} & 0 & 0 \\ \bar{b}_1 & 1 & 0 \\ \bar{b}_2 & 0 & \bar{\alpha} \end{pmatrix}$$

at $\varphi^T(p)$, we find that

$$D\varphi^{T+S}(0,0,s) = \begin{pmatrix} * & 0 & 0 \\ \bar{b}_1\alpha^{-1} + b_1 & 1 & 0 \\ * & 0 & * \end{pmatrix}.$$

Using $\bar{b}_1(0) = 0$ this gives

$$\begin{aligned} K(p, T+S) = \frac{d}{ds}(\bar{b}_1\alpha^{-1} + b_1)\big|_{s=0} &= (\frac{d}{ds}\bar{b}_1(0))\alpha^{-1}(0) + \frac{d}{ds}b_1(0) \\ &= \bar{b}_1'(0) + b_1'(0) = K(\varphi^T(p), S) + K(T, p), \end{aligned}$$

as required. □

Lemma 3.3. *The cohomology class of the longitudinal KAM-cocycle is unaffected by coordinate changes.*

Proof. Consider coordinate changes to coordinates that also have our desired list of properties. To see how the longitudinal KAM-cocycle changes we examine the change in the differential of φ^T entailed by the coordinate change. We need only study points on the stable leaf. To do the coordinate calculations we agree that the coordinate change transforms variables $(\tilde{u}, \tilde{t}, \tilde{s})$ to (u, t, s). Variables in coordinates at $\varphi^T(p)$ are marked by a subscript T. At a point $(0, 0, \tilde{s})$ an allowed coordinate change has differential

$$\begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ * & 0 & a^{-1} \end{pmatrix}$$

and the inverse in coordinates at $\varphi^T(p)$ is

$$\begin{pmatrix} a_T^{-1} & 0 & 0 \\ -a_T^{-1}b_T & 1 & 0 \\ * & 0 & a_T, \end{pmatrix}$$

with entries evaluated at $\tilde{s}_T$. Note that $a = \dfrac{d\tilde{s}}{ds}$. In these new coordinates the differential of φ^T at $(0,0,\tilde{s})$ becomes

$$\begin{pmatrix} a_T^{-1} & 0 & 0 \\ -a_T^{-1}b_T & 1 & 0 \\ * & 0 & a_T \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ b_1 & 1 & 0 \\ b_2 & 0 & \alpha \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ * & 0 & a^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} * & 0 & 0 \\ -a\alpha^{-1}a_T^{-1}b_T + ab_1 + b & 1 & 0 \\ * & * & a^{-1}\alpha a_T \end{pmatrix}.$$

Note that therefore $\dfrac{d\tilde{s}_T}{d\tilde{s}} = a^{-1}\alpha a_T$.

This gives

(3.1)

$$\tilde{K}(p,T) := \frac{d}{d\tilde{s}}\tilde{b}_1|_{\tilde{s}=0} = -a(0)\alpha^{-1}(0)a_T^{-1}(0)\frac{d}{d\tilde{s}}b_T(0) + a(0)\frac{d}{d\tilde{s}}b_1(0) + \frac{d}{d\tilde{s}}b(0)$$
$$= \frac{d}{ds}b_1(0) + \frac{d}{d\tilde{s}}b(0) - \frac{d\tilde{s}}{d\tilde{s}_T}\frac{d}{d\tilde{s}}b_T(0) = K(p,T) + b'(0) - b_T'(0),$$

which is cohomologous to K. □

Next we show that the longitudinal KAM-cocycle is indeed an obstruction to the regularity of $E^u \oplus E^s$ being any higher than Zygmund.

Proposition 3.4. *If $E^u \oplus E^s$ is "little Zygmund" (Definition 1.2) then the longitudinal KAM-cocycle is null cohomologous.*

Proof. If $E^u \oplus E^s$ is "little Zygmund" then the modulus of continuity is $o(|x \log|x||)$. Let p be any T-periodic point. Then in our usual coordinates (2.1) gives

$$e(s_\varphi) = \alpha(s)(b_1(s) + e(s)) = \alpha(0)(1 + o(s))(b_1'(0)s + o(s) + e(s)).$$

Since $\alpha(s) = \alpha(0)(1 + o(s))$ and $s_\varphi = \alpha(0)s + O(s^2)$ we find, writing $\alpha := \alpha(0)$, that

$$e(\alpha s) = e(s_\varphi)(1 + o(s^2 \log|s|))$$
$$= \alpha(1 + o(s))(b_1'(0)s + o(s) + e(s)) = \alpha(K(p,T)s + e(s) + o(s)).$$

Recursively, this gives

$$e(\alpha^n s) = \alpha^n (nK(p,T)s + e(s) + \sum_{i=0}^{n-1} \alpha^{1-i} o(\alpha^i s)).$$

Since the terms of the sum converge to 0 we get $\sum_{i=0}^{n-1} \alpha^{1-i} o(\alpha^i s)/n \to 0$ as $n \to \infty$. Therefore,

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{e(\alpha^n s)}{\alpha^n s \log(\alpha^n s)} \\
&= \lim_{n\to\infty} \Big[\frac{nK(p,T)}{n\log\alpha + \log s} + \frac{e(s)}{\alpha^n s\log(\alpha^n s)} + \frac{\sum_{i=0}^{n-1}\alpha^{1-i}o(\alpha^i s)}{s\log(\alpha^n s)}\Big] = \frac{K(p,T)}{\log\alpha},
\end{aligned}
$$

so $K(p,T) = 0$. □

4. Vanishing of the obstruction

We now begin to study what happens when the longitudinal KAM-cocycle is trivial. As a first step we show that this allows more perfectly adapted coordinate systems.

Lemma 4.1. *If the longitudinal KAM-cocycle is a coboundary then there are nonstationary local coordinates in which it vanishes identically.*

Proof. If the longitudinal KAM-cocycle is null-cohomologous then there is a smooth $k\colon M \to \mathbb{R}$ such that $K(p,T) = k(\varphi^T(p)) - k(p)$ for all $p \in M$, $T \in \mathbb{R}$. With the notations from the proof of Lemma 3.3 define a coordinate change at p by

$$
\begin{pmatrix} u \\ t \\ s \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{t} + k(p)\tilde{u}\tilde{s} \\ \tilde{s} \end{pmatrix}.
$$

In these new coordinates $\tilde{K}(p,T) = 0$ for all $p \in M$, $t \in \mathbb{R}$ by (3.1). □

The first direct consequence of the existence of such coordinates is that in this case $E^U \oplus E^s$ is Lipschitz continuous.

Proposition 4.2. *If the longitudinal KAM-cocycle is a coboundary then $E^u \oplus E^s$ is Lipschitz continuous.*

Proof. We combine the arguments of Proposition 2.2 and Proposition 2.3, using that in our new coordinates the cocycle is trivial, i.e., $b_1'(0) = 0$ in (2.1), so $b_1(s) = O(s^2)$. Assume that $|e(s)| \le K|s|$ with uniform K. Then the first calculation in Proposition 2.2 becomes

$$
\begin{aligned}
|e(s_\varphi)| = |\alpha(s)||b_1(s) + e(s)| &\le |\alpha(s)|(O(s^2) + K|s|) \\
&= (K + O(s))|\alpha(s)s| = K(1+\kappa(s))|s_\varphi|,
\end{aligned}
$$

with $\kappa(s) = O(s)$ decreasing in K. Together with the argument from Proposition 2.2 for $\bar{e}$, which works for $H = 1$, this implies Lipschitz continuity as in Corollary 2.4. □

5. SMOOTHNESS

We now show that Lipschitz continuity implies smoothness.

Proposition 5.1. *If the volume preserving Anosov flow is C^k and the subbundle $E^u \oplus E^s$ is Lipschitz continuous, then $E^u \oplus E^s$ is C^{k-1}.*

Proof. By the Livshitz homological theorem [L] there is a C^k invariant volume form Ω. The canonical invariant one-form A associated to the flow by

$$A(\dot{\varphi}) = 1, \quad A(E^u \oplus E^s) = 0$$

is Lipschitz continuous, and we aim to prove that it is C^{k-1}.

In local charts as in Lemma 2.1 we have

$$A = dt + \lambda(u,s)ds + \beta(u,s)du,$$

where λ and β are Lipschitz-continuous functions independent of the variable t. Then the Anosov vector field $X = \dot{\varphi}\,(= \partial_t)$ is in the kernel of the integrable 2-form dA and the L^1 3-form $A \wedge dA$ is also flow invariant. By ergodicity there is a $K \in \mathbb{R}$ such that

$$A \wedge dA \overset{\text{a.e.}}{=} K\Omega,$$

with respect to the invariant measure associated to Ω. Equivalently, $dA \overset{\text{a.e.}}{=} w$, with $w = K\Omega(X, \cdot)$ a C^k invariant 2-form. We claim that w is exact.

We use that the classical Stokes Theorem holds for Lipschitz continuous forms [P]: Let c_θ, $\theta \in [-\epsilon, \epsilon]$ be a family of regular disjoint 2-cycles such that $[c_\theta] = c \in H^2(M)$ and set $\Delta = \cup_{t\in[-\epsilon,\epsilon]} c_\theta$. By the Fubini theorem

$$2\epsilon \cdot \omega(c) = \int_{-\epsilon}^{\epsilon} \Big(\int_{c_\theta} w \Big) d\theta = \int_\Delta d\theta \wedge w = \int_\Delta d\theta \wedge dA = \int_{-\epsilon}^{\epsilon} \Big(\int_{c_\theta} dA \Big) d\theta = 0,$$

and w is exact.

Therefore, there is a C^k one-form $\tilde{A}$ such that $w = d\tilde{A}$. Then the Lipschitz-continuous 1-form $A - \tilde{A}$ is almost everywhere closed. Choose a reference Riemannian metric g. Using the same type of Fubini-Stokes argument there exist a g-harmonic 1-form H in the same cohomology class and a 1-form μ such that

$$A - \tilde{A} = H + \mu, \quad d\mu \overset{\text{a.e.}}{=} 0, \quad \int_\gamma \mu = 0$$

for any 1-cycle γ. We aim to show that μ is exact, but instead of using the same trick as before we will be more explicit to get higher regularity.

In adapted local coordinates at $p \in M$ denote by $[0,x]$, $x = (u,t,s)$, the image of the segment by the corresponding local chart. Introduce for any $c \in \mathbb{R}$ a local Lipschitz contiuous function

$$f_p^c(x) := c + \int_{[0,x]} \mu.$$

For every x, v and small ϵ the Stokes Theorem gives

$$f_p^c(x+\epsilon v) - f_p^c(x) = \int_{[x,x+\epsilon v]} \mu - \int_{T_\epsilon} d\mu,$$

where T_ϵ is the oriented triangle $(0, x, x+\epsilon v)$. Then $df_p^c(x)(v) = \mu(x)(v)$ for almost every x and v. Since the integral of μ along any closed curve vanishes there is a global function f that locally coincides with one of the functions f_p^c, *i.e.*,

$$A \overset{\text{a.e.}}{=} \tilde{A} + H + df. \tag{5.1}$$

We conclude the proof by showing that df is C^{k-1}.

Using the Anosov vector field and the definition of the canonical 1-form A we write

$$1 - \tilde{A}(X) - H(X) \overset{\text{a.e.}}{=} df(X).$$

The terms on the left are C^k, so by the measurable version of the Livshitz Theorem there exists a C^k boundary $b\colon M \to \mathbb{R}$ such that

$$1 - \tilde{A}(X) - H(X) = db(X).$$

This in particular means that the Lipschitz-continuous function $\rho := b - f$ satisfies

$$d\rho(X) \overset{\text{a.e.}}{=} 0. \tag{5.2}$$

We show that this implies flow-invariance of ρ. Choose $p \in M$ and an adapted chart in which the Anosov vector field X is expressed as ∂_t. Assume there are $x_0 = (u_0, t_0, s_0)$ and $t \in \mathbb{R}$ such that $\rho(\varphi_t(x_0)) - \rho(x_0) \neq 0$. By continuity there is an open set U in the transversal $\{(u, t_0, s)\}$ such that

$$0 < \int_U (\rho(\varphi_t(u,t_0,s)) - \rho(u,t_0,s))\,du\,ds = \int_U \Big(\int_{t_0}^{t_0+t} d\rho(X(w))dw \Big)\,duds,$$

contrary to (5.2).

Invariance and ergodicity imply that ρ is constant (everywhere by continuity), which implies $db = df$. This concludes the proof: We have shown that $A = \tilde{A} + H + db$ because both sides of (5.1) are continuous. Thus $A \in C^{k-1}$ and so its kernel $E^u \oplus E^s$ is C^{k-1}. □

By a theorem of Plante [Pl], invariance of such a smooth form implies the dichotomy in 5. of Theorem 1.3. This completes the proof of Theorem 1.3

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