JOURNAL OF MODERN DYNAMICS
VOLUME 4, NO. 3, 2010, 549–569

doi: 10.3934/jmd.2010.4.549

ZYGMUND STRONG FOLIATIONS IN HIGHER DIMENSION

YONG FANG, PATRICK FOULON AND BORIS HASSELBLATT
(Communicated by Anatole Katok)

ABSTRACT. For a compact Riemannian manifold M, $k \geq 2$ and a uniformly quasiconformal transversely symplectic C^k Anosov flow $\varphi\colon \mathbb{R}\times M \to M$ we define the *longitudinal KAM-cocycle* and use it to prove a rigidity result: $E^u \oplus E^s$ is Zygmund-regular, and higher regularity implies vanishing of the longitudinal KAM-cocycle, which in turn implies that $E^u \oplus E^s$ is Lipschitz-continuous. Results proved elsewhere then imply that the flow is smoothly conjugate to an algebraic one.

1. INTRODUCTION

1.1. **Statement of main result.** For Anosov systems (Definition 1.6) with both continuous and discrete time, interesting phenomena of smooth and geometric rigidity have been observed in connection with the degree of (transverse) regularity of the (weak) stable and unstable subbundles of these systems. The seminal result was the study of volume-preserving Anosov flows on 3-manifolds by Hurder and Katok [15], which showed that the weak-stable and weak-unstable foliations are $C^{1+\text{Zygmund}}$ (Definition 1.7) and that there is an obstruction to higher regularity whose vanishing implies smoothness of these foliations. This, in turn, happens only if the Anosov flow is smoothly conjugate to an algebraic one. The cocycle obstruction described by *K*atok and Hurder was first observed by *A*nosov and is the first nonlinear coefficient in the *M*oser normal form. Therefore one might call it the *KAM*-cocycle. This should not be confused with "KAM" as in "Kolmogorov–Arnold–Moser", and Hurder and Katok refer to this object as the *Anosov-cocycle*.

In [7], we showed some analogous rigidity features associated with the *longitudinal* direction, *i.e.,* associated with various degrees of regularity of the sum of the *strong* stable and unstable subbundles: For a volume-preserving Anosov flow on a 3-manifold the strong stable and unstable foliations are Zygmund-regular [25, Chapter II, Equation (3·1)], and there is an obstruction to higher regularity, which admits a direct geometric interpretation and whose vanishing

Received May 27, 2010; revised June 27, 2010.
2000 *Mathematics Subject Classification:* Primary: 37D20, 37D40; Secondary: 53C24, 53D25.
Key words and phrases: Anosov flow, strong invariant subbundles, quasiconformal, Zygmund regularity, smooth rigidity, geometric rigidity, foliations.

implies high smoothness of the joint strong subbundle and that the flow is either a suspension or a contact flow. We now push this to higher-dimensional systems:

THEOREM 1.1. *Let M be a compact Riemannian manifold of dimension at least 5, $k \geq 2$, $\varphi\colon \mathbb{R} \times M \to M$ a uniformly quasiconformal transversely symplectic C^k Anosov flow.*

Then $E^u \oplus E^s$ is Zygmund-regular and there is an obstruction to higher regularity that defines the cohomology class of a cocycle we call the longitudinal KAM-cocycle. *This obstruction can be described geometrically as the curvature of the image of a transversal under a return map, and the following are equivalent:*

(a) *$E^u \oplus E^s$ is "little Zygmund" (see Definition 1.7).*
(b) *The longitudinal KAM-cocycle is a coboundary.*
(c) *$E^u \oplus E^s$ is Lipschitz continuous.*
(d) *φ is up to finite covers, constant rescaling and a canonical time-change C^k conjugate to the suspension of a symplectic Anosov automorphism of a torus or the geodesic flow of a real hyperbolic manifold.*

This result implies in particular that almost all time-changes of a geodesic flow of a hyperbolic manifold (namely, all noncanonical ones) produce a flow with nontrivial KAM-cocycle. In a similiar light, Theorem 1.9 can be viewed as saying that most 3-dimensional magnetic Anosov flows have nontrivial longitudinal KAM-cocycles. However, combining uniform quasiconformality with being magnetic rules this out: Geodesic flows of hyperbolic metrics are rigid in the quasiconformal category even among magnetic flows (Definition 1.4).

THEOREM 1.2. *Let (N, g) be a n-dimensional closed negatively curved Riemannian manifold and Ω a C^∞ closed 2-form of N. For $\lambda \in \mathbb{R}$ small, let φ^λ be the magnetic Anosov flow of the pair $(g, \lambda\Omega)$. Suppose that $n \geq 3$ and φ^λ is uniformly quasiconformal. Then g has constant negative curvature and $\lambda\Omega = 0$. In particular, the longitudinal KAM-cocycle of φ^λ is a coboundary.*

Magnetic flows are introduced below; for the moment it may suffice to remark that the magnetic flow for $\lambda = 0$ is the geodesic flow.

1.2. **Background and terminology.** We now introduce the notions that play a role in this result and the proof.

1.2.1. *Transversely symplectic flows.*

DEFINITION 1.3. Let φ be a C^∞ flow on a closed manifold M. Denote by X the generating vector field of φ. The flow φ is said to be *transversely symplectic* if there exists a C^∞ closed 2-form ω on M such that $\operatorname{Ker}\omega = \mathbb{R}X$. The closed 2-form ω is said to be the *transverse symplectic form* of φ. It is easy to see that ω is φ-invariant.

Examples of transversely symplectic flows can be obtained via the following classical construction: Let (N, ω^N) be a C^∞ symplectic manifold and $H\colon N \to \mathbb{R}$

a C^∞ Hamiltonian function. The Hamiltonian flow φ^N of H with respect to ω^N is generated by the vector field X^N determined by $i_{X^N}\omega^N = -dH$. Observe that φ^N preserves all the energy levels of H. Therefore, if the function H is proper and $a \in \mathbb{R}$ is a regular value of H, then the energy level $H^{-1}(a)$ is a closed φ^N-invariant C^∞ submanifold of N. Let $M = H^{-1}(a)$ and $\varphi = \varphi^N\restriction_M$. It is easy to verify that φ is transversely symplectic with respect to the closed 2-form $\omega^N\restriction_M$.

1.2.2. *Magnetic flows.* Magnetic flows are important examples of transversely symplectic flows, which are constructed as follows:

DEFINITION 1.4. Let (N, g) be a closed C^∞ Riemannian manifold and Ω a C^∞ closed 2-form of N. Let α denote the C^∞ 1-form on TN obtained by pulling back the Liouville 1-form of T^*N via the Riemannian metric. For $\lambda \in \mathbb{R}$, the *twisted symplectic structure* ω_λ is defined as

$$\omega_\lambda = d\alpha - \lambda\pi^*\Omega,$$

where $\pi\colon TN \to N$ denotes the canonical projection. Let $H\colon TN \to \mathbb{R}$ be the Hamiltonian function defined as

$$H(v) = \tfrac{1}{2}\, g(v, v)$$

for any $v \in TN$. The energy level $H^{-1}(1/2)$ is the unit sphere bundle SN. Let φ^λ be the restriction to SN of the Hamiltonian flow of H with respect to ω_λ. It is clear that φ^λ is a transversely symplectic flow with respect to $\omega_\lambda\restriction_{SN}$, which is said to be the *magnetic flow* of the pair $(g, \lambda\Omega)$.

REMARK 1.5. If the Riemannian metric g is negatively curved and λ is small enough, then the magnetic flow φ^λ is a transversely symplectic Anosov flow.

To see that ω_λ is a symplectic form note that TN splits into the horizontal and vertical subbundles $H(TN)$ and $V(TN)$ of the Riemannian metric g, and $d\alpha$ vanishes on either of these. If $v \in V(TM)$, then $D\pi(v) = 0$, so for $w \in TM$ we have $\pi^*\Omega(v, w) = 0$. Since $d\alpha$ is symplectic, for any $u \in H(TN)$ (respectively, $u \in V(TN)$), there exists $v \in V(TN)$ (respectively, $v \in \mathbb{H}(TN)$) such that $0 \neq d\alpha(u, v) = \omega_\lambda(u, v)$. Therefore, ω_λ is a symplectic form.

1.2.3. *Anosov flows and time-changes.*

DEFINITION 1.6 ([17])**.** Let M be a manifold and $\varphi\colon \mathbb{R} \times M \to M$ be a smooth flow. Then φ is said to be an Anosov flow if the tangent bundle TM splits as $TM = E^\varphi \oplus E^u \oplus E^s$ (the *flow*, *strong-unstable* and *strong-stable directions*, respectively), where $E^\varphi(x) = \mathbb{R}\dot\varphi(x) \neq \{0\}$ for all $x \in M$, in such a way that there are constants $C > 0$ and $\eta > 1 > \lambda > 0$ such that for $t > 0$ we have

$$\left\| D\varphi^{-t}\restriction_{E^u} \right\| \le C\eta^{-t} \quad \text{and} \quad \left\| Df^t\restriction_{E^s} \right\| \le C\lambda^t.$$

The *weak-unstable* and *weak-stable* directions are $E^\varphi \oplus E^u$ and $E^\varphi \oplus E^s$, respectively.

A *canonical time-change* is defined using a closed 1-form β by replacing the generator X of the flow by the vector field $X/(1+\beta(X))$, provided β is such that the denominator is positive. (See Section 2 for more on canonical time-changes.)

1.2.4. *Regularity.* The subbundles are invariant and (Hölder-) continuous with smooth integral manifolds W^u and W^s that are coherent in that $q \in W^u(p) \Rightarrow W^u(q) = W^u(p)$. W^u and W^s define laminations (continuous foliations with smooth leaves).

DEFINITION 1.7. We say that a function f between metric spaces is *Hölder-continuous* if there is an $H > 0$, the *Hölder exponent*, such that $d(f(x), f(y)) \le \text{const.} d(x, y)^H$ whenever $d(x, y)$ is sufficiently small. We specify the exponent by saying that a function is H-Hölder. A continuous function $f: U \to L$ on an open set $U \subset L'$ in a normed linear space to a normed linear space is said to be *Zygmund-regular* if there is $Z > 0$ such that $\|f(x+h)+f(x-h)-2f(x)\| \le Z\|h\|$ for all $x \in U$ and sufficiently small $\|h\|$. To specify a value of Z we may refer to a function as being Z-Zygmund. The function is said to be *"little Zygmund"* (or "zygmund") if $\|f(x+h)+f(x-h)-2f(x)\| = o(\|h\|)$. For maps between manifolds these definitions are applied in smooth local coordinates (see Appendix A).

Zygmund regularity implies modulus of continuity $O(|x \log|x||)$ and hence H-Hölder-continuity for all $H < 1$ [25, Chapter II, Theorem (3·4)]. It follows from Lipschitz-continuity and hence from differentiability. Being twice differentiable implies a quadratic Zygmund condition: $\|f(x+h)+f(x-h)-2f(x)\| = O(\|h\|^2)$. Being "little Zygmund" implies modulus of continuity $o(|x \log|x||)$ and follows from differentiability but not from Lipschitz-continuity.

The regularity of the unstable subbundle E^u is usually substantially lower than that of the weak-unstable subbundle $E^u \oplus E^\varphi$. The exception are geodesic flows, where the strong unstable subbundle is obtained from the weak-unstable subbundle by intersecting with the kernel of the invariant contact form. This has the effect that the strong-unstable and weak-unstable subbundles have the same regularity. However, time changes affect the regularity of the strong-unstable subbundle, and this is what typically keeps its regularity below C^1. In [7] we presented a *longitudinal KAM-cocycle* that is the obstruction to differentiability, and we derived higher regularity from its vanishing.

THEOREM 1.8 ([7, Theorem 3])**.** *Let M be a* 3*-manifold, $k \ge 2$, $\varphi: \mathbb{R} \times M \to M$ a C^k volume-preserving Anosov flow. Then $E^u \oplus E^s$ is Zygmund-regular, and there is an obstruction to higher regularity that can be described geometrically as the curvature of the image of a transversal under a return map. This obstruction defines the cohomology class of a cocycle (the longitudinal KAM-cocycle), and the following are equivalent:*

(a) *$E^u \oplus E^s$ is "little Zygmund" (see Definition 1.7).*
(b) *The longitudinal KAM-cocycle is a coboundary.*
(c) *$E^u \oplus E^s$ is Lipschitz.*
(d) $E^u \oplus E^s \in C^{k-1}$.
(e) *φ is a suspension or contact flow.*

In (e) no stronger rigidity should be expected because $E^u \oplus E^s$ is smooth for all suspensions and contact flows. In [1, 20] this was applied to magnetic flows.

THEOREM 1.9 ([1]). *Let (Σ, g) be a closed oriented Riemannian surface and Ω a closed 2-form of Σ. For $\lambda \in \mathbb{R}$, suppose that the magnetic flow φ^λ on $S\Sigma$ is Anosov.*

(a) *If Ω is exact, then the longitudinal KAM-cocycle of φ^λ is a coboundary if and only if $\lambda\Omega \equiv 0$.*
(b) *If Ω is not exact, then the longitudinal KAM-cocycle of φ^λ is a coboundary if and only if g has constant negative curvature and Ω is a constant multiple of the area form.*

1.2.5. *Uniform quasiconformality.* The work by Hurder and Katok in [15] inspired developments of substantial extensions to higher dimensions, see, for example, [14]. The present work extends our earlier work to higher-dimensional systems in this "longitudinal" context. This requires somewhat stringent assumptions, however.

DEFINITION 1.10. An Anosov flow is said to be *uniformly quasiconformal* if

$$K_i(x,t) := \frac{\| d\varphi^t \restriction_{E^i} \|}{\| d\varphi^t \restriction_{E^i} \|^*} \tag{1}$$

is bounded on $\{u,s\} \times M \times \mathbb{R}$, where $\|A\|^* := \min_{\|v\|=1} \|Av\|$ is the conorm of a linear map A.

A uniformly quasiconformal Anosov flow is said to be *rate-symmetric* if it preserves volume and the stable and unstable subbundles have the same dimension.

REMARK 1.11. We use the term rate-symmetric because for quasiconformal volume-preserving Anosov flows whose stable and unstable subbundles have the same dimension the contraction and expansion rates are close to being reciprocal. This condition follows from transverse symplecticity, which is a hypothesis for our main result. However, some of our results use only rate-symmetry. Note that rate-symmetry implies that at a periodic point the restriction to a stable or unstable subspace of the differential of a return map is complex-diagonalizable with all eigenvalues on the same circle.

1.3. **Rigidity.** In the 3-dimensional case we showed that smoothness of $E^u \oplus E^s$ implies that φ is a suspension or contact flow, but in the present situation we obtain more detailed information because we assume quasiconformality and can apply the following rigidity theorem:

THEOREM 1.12 ([3, Corollary 3]). *Let M be a compact Riemannian manifold and $\varphi\colon \mathbb{R} \times M \to M$ a transversely symplectic Anosov flow with* $\dim E^u \geq 2$ *and* $\dim E^s \geq 2$. *Then φ is quasiconformal if and only if φ is up to finite covers C^∞ orbit equivalent either to the suspension of a symplectic hyperbolic automorphism of a torus, or to the geodesic flow of a closed hyperbolic manifold.*

This also serves to illustrate that the assumption of uniform quasiconformality is quite restrictive. We should also point out that our result about rigidity of the situation with $E^u \oplus E^s \in C^1$ overlaps with a closely related earlier one, although the proof is independent:

THEOREM 1.13 ([3, Corollary 2]). *Let φ be a C^∞ volume-preserving quasiconformal Anosov flow. If $E^s \oplus E^u \in C^1$ and $\dim E^u \geq 3$ and $\dim E^s \geq 2$ (or, $\dim E^s \geq 3$ and $\dim E^u \geq 2$), then φ is up to finite covers and a constant change of time scale C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to a canonical time change (Definition 1.6) of the geodesic flow of a closed hyperbolic manifold.*

The implication "(c) ⇒ (d)" in Theorem 1.1 is a consequence of Theorem 1.12 and the following results from [8].

THEOREM 1.14 ([8, Corollary 1.4]). *Let M be a compact locally symmetric space with negative sectional curvature and consider a time-change of the geodesic flow whose canonical 1-form is Lipschitz-continuous. Then the canonical form of the time-change is C^∞, and the time-change is canonical (Definition 1.6).*

THEOREM 1.15 ([8, Corollary 1.8]). *Let ψ be a hyperbolic automorphism of a torus or a nilmanifold $\Gamma\backslash M$ and consider a time-change of the suspension whose canonical 1-form is Lipschitz-continuous. Then the canonical form of the time-change is C^∞, and the time-change is a canonical time-change (Definition 1.6).*

1.4. **Structure of this paper.** We first make a few remarks about canonical time-changes because not all readers may be familiar with these. After some preliminary developments pertinent to establishing regularity of the strong subbundles in the subsequent two sections, we then establish Zygmund-regularity of the joint strong subbundles in Section 5.

The two sections that follow establish (a) ⇒ (b) ⇒ (c) in Theorem 1.1 (while (c) ⇒ (d) was already discussed in the previous section). Specifically, (a) ⇒ (b) is Proposition 6.5 and (b) ⇒ (c) is Proposition 7.2.

Section 8 establishes Theorem 1.2, and the paper concludes with an appendix to show that Zygmund-regularity is well-defined when one uses charts; the account here is based on personal communication with Rafael de la Llave, and we are grateful for his permission to reproduce his arguments here.

2. CANONICAL TIME-CHANGES

We make a few remarks here about canonical time-changes (Definition 1.6) that are not needed for the sequel but may serve to show why this is a natural class of time-changes to expect in rigidity results for flows.

PROPOSITION 2.1 (Trivial time-changes). *Consider a flow φ^t with generating vector field X and a smooth function $f\colon M \to \mathbb{R}$ such that $1 + df(X) > 0$. Then, $\Psi\colon x \mapsto \varphi^{f(x)}(x)$ conjugates the flow generated by the vector field $X_f := \frac{X}{1+df(X)}$ to φ^t.*

Proof. Smoothness of f and $1+df(X)>0$ ensure that Ψ is a diffeomorphism. Now we write $x_t=\varphi^t(x)$ and use the chain rule to compute

$$\begin{aligned} d\Psi(X(x)) = \frac{d\Psi}{dt} = \frac{d}{dt}\varphi^{f(x_t)}(x_t)\big|_{t=0} &= \frac{d\varphi}{dt}\Big|_{t=0} df(X)(x) + X\big(\varphi^{f(x)}(x)\big) \\ &= X\big(\varphi^{f(x)}(x)\big)\cdot df(X)(x) + X\big(\varphi^{f(x)}(x)\big) \\ &= (1+df(X)(x))X\big(\varphi^{f(x)}(x)\big), \end{aligned}$$

which gives $d\Psi(X_f)=X$ upon division by $1+df(X)(x)$. □

PROPOSITION 2.2 (Cohomology class)**.** *If α and β are cohomologous closed 1-forms with $1+\alpha(X)>0$ and $1+\beta(X)>0$ then the associated canonical time-changes of X are smoothly conjugate.*

REMARK 2.3. This tells us that the cohomology class of α is the material ingredient in a canonical time-change by α.

Proof. Writing $\beta=\alpha+df$ with smooth f we observe that

$$\frac{X}{1+\beta(X)} = \frac{X}{1+\alpha(X)+df(X)} = \frac{\frac{X}{1+\alpha(X)}}{1+df\big(\frac{X}{1+\alpha(X)}\big)} = \left(\frac{X}{1+\alpha(X)}\right)_f.$$

Now use Proposition 2.1. □

PROPOSITION 2.4 (Regularity)**.** *Suppose X_0 generates an Anosov flow, and α is a closed 1-form such that $1+\alpha(X_0)>0$. If A_0 denotes the canonical form for X_0 then $A:=A_0+\alpha$ is the canonical form for $X:=X_0/(1+\alpha(X_0))$.*

REMARK 2.5. In particular, this shows that canonical time-changes with smooth closed forms do not affect the regularity of the canonical form.

Proof. We first note that two invariant 1-forms for an Anosov flow are proportional: Both being constant on X, this follows from the fact that a continuous 1-form that vanishes on X is trivial [6, Lemma 1].

Since α is closed we have $dA=dA_0$. Also

$$A(X) = \frac{A_0(X_0)+\alpha(X_0)}{1+\alpha(X_0)} = 1,$$

which implies that $\mathcal{L}_X A=0$, *i.e.,* A is X-invariant and hence proportional to the canonical 1-form of X. But $A(X)=1$ then implies that A is equal to the canonical 1-form of X. □

3. LOCAL ADAPTED COORDINATES

To prove regularity of $E^u\oplus E^s$ we use the Hadamard graph transform method [11, 17]. It was developed in order to prove the existence of invariant manifolds, and we examine it with a view to regularity of the subbundles. In essence, we apply it to the one-form whose kernel is $E^u\oplus E^s$.

The graph transform acts on subbundles by $\varphi^t_* E(p) := D\varphi^t\big(E(\varphi^{-t}(p))\big)$, *i.e.,* the subbundle E is acted on by the differential of φ^t. If one considers the space of continuous subbundles moderately close to E^u with the distance defined by (the supremum of) pointwise angles, then the graph transform is a contracting map and hence has a unique fixed point, E^u. In order to prove regularity of E^u it therefore suffices to consider the same transformation in a complete subspace of subbundles of the desired regularity, and to prove that the orbit of some initial subbundle remains in that space. This proves that the fixed point E^u is in that space as well.

In fact, there is a little less to show than it would appear: The subbundle E^u is smooth in the flow direction by invariance, and it is C^{k-1} along W^u because W^u has C^k leaves. Therefore it suffices to show the desired regularity along W^s. To set up and illustrate the method we do this here for Hölder-continuity. Achieving Zygmund regularity requires a little more preparation, which we carry out in the next section.

Take $T \geq 1$, fixed for now. After possibly rescaling time we will from now assume that φ^1 contracts stable manifolds and expands unstable manifolds. By adjusting multiplicative constants, the estimates we get for $T \geq 1$ can be extended to $T \in [0,1)$.

We consider the graph transform φ_*^T henceforth. The next lemma refers to *local* stable and unstable manifolds. The local unstable manifold $W^u_{\text{loc}}(p)$ of a point p is defined to be the connected component containing p of the intersection of $W^u(p)$ with an ϵ-neighborhood of p.

LEMMA 3.1. *There exist local coordinates adapted to the invariant laminations,* i.e., *coordinate systems* $\Psi\colon M \times (-\epsilon,\epsilon)^{2n+1} \to M$ *such that* $\Psi_p := \Psi(p,\cdot)$ *satisfies the following.*

(a) Ψ_p *is a* C^k*-diffeomorphism onto a neighborhood of* p *for every* $p \in M$.
(b) Ψ_p *depends continuously/Hölder-continuously/Zygmund-continuously on* p *if the strong stable and unstable laminations do.*
(c) Ψ_p *preserves volume for each* $p \in M$*; if* φ^t *is transversely symplectic then* Ψ_p *sends the standard symplectic structure to the one on transversals in* M.
(d) $\Psi_p(0) = p$.
(e) $\Psi_p((-\epsilon,\epsilon)^n \times \{0\} \times \{0\}) = W^u_{\text{loc}}(p) \cap \Psi_p\big((-\epsilon,\epsilon)^{2n+1}\big)$.
(f) $\Psi_p^{-1}\big(\varphi^\delta(\Psi_p(u,t,s))\big) = (u, t+\delta, s)$ *for* $|\delta| < \epsilon$.
(g) $\Psi_p(\{0\} \times \{0\} \times (-\epsilon,\epsilon)^n) = W^s_{\text{loc}}(p) \cap \Psi_p\big((-\epsilon,\epsilon)^{2n+1}\big)$.

REMARK 3.2. We will denote by (M,Ψ) a manifold together with such a choice of charts.

Such coordinates are fairly standard [17, Section 6.4b], [12, Proposition 3.1]. Even more sophisticated adaptations are possible [2, 15], and we presently produce one such (Theorem 3.4). One basic ingredient is Moser's homotopy trick [17, Section 5.1e], refined to impose conditions beyond volume normalization. It is natural to denote the coordinate variables by (u,t,s).

Referring to (1), we will take

$$K \geq \max\{K_i(x,t) \mid x \in M,\ t \in \mathbb{R},\ i = u,s\}, \tag{2}$$

so $\|d\varphi^t\restriction_{E^i}\| \leq K\|d\varphi^t\restriction_{E^i}\|^*$ in these coordinate charts for all $x \in M$, $t \in \mathbb{R}$, and $i = u,s$.

The promised adaptation of the coordinates uses a result by Sadovskaya obtained by refining normal-form methods of Katok and Guysinsky [10, 9, 18]. We provide here the flow-version from [4].

THEOREM 3.3 ([23, Propositions 3.3, 4.1],[4, p. 1940])**.** *Let φ^t be a uniformly quasiconformal Anosov flow on a compact manifold M and let W be a continuous invariant foliation with C^∞ leaves such that $\|d\varphi^t\restriction_{TW}\| < 1$ for $t > 0$. Then for any $x \in M$ there exists a C^∞ diffeomorphism $h_x\colon W(x) \to T_xW$ such that*

(a) $h_{\varphi^t(x)} \circ \varphi^t = d\varphi^t_x \circ h_x$,
(b) $h_x(x) = 0$ *and* $(dh_x)_x$ *is the identity map,*
(c) h_x *depends Hölder-continuously resp. Zygmund-continuously on* x *in the* C^∞ *topology if the leaves of the foliation do.*

The regularity assertion (c) can be seen from the proof in [23]. Also, a far less restrictive assumption than quasiconformality is quite sufficient here.

We now extend these linearizing coordinates from a local leaf to a neighborhood.

THEOREM 3.4. *For a rate-symmetric Anosov flow there exist coordinates such as in Lemma 3.1 with the additional property that in these coordinates the flow acts linearly on stable leaves, and these coordinates can be chosen to depend continuously/Hölder-continuously/Zygmund-continuously on the base point if the leaves of the strong stable foliation do. Moreover, if the flow is transversely symplectic, one can choose the coordinates in such a way as to send the transverse symplectic structure to the standard one.*

Proof. Start with coordinates as in Lemma 3.1. With respect to these, the linearization in Theorem 3.3 on each strong stable leaf is isotopic to the identity for sufficiently small ϵ by Theorem 3.3(b): We can write the coordinate change as $s_i' = s_i + \sum_{kl} g_{kl}(s)s_k s_l$, and for sufficiently small ϵ, $\sum_{kl} g_{kl}(s)s_k s_l$ is a contraction. But then so is $t \cdot \sum_{kl} g_{kl}(s)s_k s_l$ for $t \in [0,1]$, which shows that $s_i' = s_i + t\sum_{kl} g_{kl}(s)s_k s_l$ is a diffeomorphism for each $t \in [0,1]$.

Thus the linearization on strong stable leaves can be written as $s' = \ell^1(s)$, where ℓ^t is a flow on each strong stable leaf. We aim to produce a coordinate change that retains the properties in Lemma 3.1; in particular, it needs to be symplectic. The time-1 map of the Hamiltonian flow of $H(u,s) := \langle \frac{\partial}{\partial t}\ell^t(s)|_{t=0}, u\rangle$ produces the desired coordinate change because it is symplectic by construction, it restricts to the linearization h_x on the stable leaf, and it does not change the other properties. $\square$

We will henceforth be using these coordinates, and we refer to them as *linearizing coordinates.*

4. THE GRAPH TRANSFORM AND HÖLDER REGULARITY

Since the subbundles are invariant under the flow, the coordinate representation of the flow preserves the axes of the local coordinate system as well as volume. Note that this flow produces maps between charts at different points.

The differential of φ^T at points of the stable leaf (the third coordinate plane, or "s-axis") therefore takes the following form:

$$D\varphi^T(0,0,s) = \begin{pmatrix} \beta_T(s) & 0 & 0 \\ \eta_T(s) & 1 & 0 \\ \xi_T(s) & 0 & \alpha_T(s) \end{pmatrix} = \begin{pmatrix} \beta & 0 & 0 \\ \eta(s) & 1 & 0 \\ \xi(s) & 0 & \alpha \end{pmatrix}, \tag{3}$$

where $\alpha(s)$, $\beta(s)$ and $\xi(s)$ are $n\times n$-matrices, $\eta(s) = O(\|s\|)$, $\xi(s) = O(\|s\|)$, and $\|\alpha(s)\| < 1$ (using a Lyapunov metric). Here, all $O(s)$ are uniform in p.

We use here that in the coordinates provided by Theorem 3.4 the entry α in (3) does not depend on s, and $\varphi^T(0,0,s) = (0,0,\alpha s)$ in local coordinates. When the flow is transversely symplectic, $\beta = {}^t\alpha^{-1}$ is also independent of s, and this is the reason for writing β instead of $\beta(s)$.

In adapted coordinates a subbundle transverse to $E^\varphi \oplus E^s$ is represented by graphs of linear maps from E^u to $E^\varphi \oplus E^s$. Using the canonical representation of the tangent bundle of $\mathbb{R}^{2n+1}$ we can write any subspace transverse to the ts-"plane" as the *image* of a linear map given by a column matrix

$$\begin{pmatrix} I \\ e \\ \bar{e} \end{pmatrix} \quad \text{or by} \quad \begin{pmatrix} I \\ e(s) \\ \bar{e}(s) \end{pmatrix},$$

when we wish to indicate the dependence on s. Note that here e is a row vector and hence represents a 1-form.

PROPOSITION 4.1. *The canonical invariant one-form A associated to the flow by*

$$A(\dot{\varphi}) = 1, \qquad A(E^u \oplus E^s) = 0$$

has the same regularity as e in the above representation of E^u.

Proof. By invariance, A is smooth in the flow direction, so we need to verify the claim along the stable manifold. To that end we show that $A_{(0,0,s)} = dt - e(s)$. Since $A_{(0,0,s)}(\partial/\partial t) = 1$, $A_{(0,0,s)}(\partial/\partial s) = 0$ and $A_{(0,0,s)}(E^u_{(0,0,s)}) = 0$, we have $A_{(0,0,s)} = dt + \sum_i f_i(0,0,s)\,du^i$. Choosing a basis $v_i = e(s)(\partial/\partial u^i)\partial/\partial t + \partial/\partial u_i$ of E^u_s we find that $0 = A_{(0,0,s)}(v_i) = e(s)(\partial/\partial u^i) + f_i(0,0,s)$ for all i. This gives $A_{(0,0,s)} = dt - e(s)$. □

To establish the desired regularity we use the Hadamard Graph-Transform [17]. This is simply the natural action of $d\varphi^t$ on subbundles transverse to $E^\varphi \oplus E^s$ given by $\varphi^T_*(F)(x) = d\varphi^t(F(\varphi^{-t}(x)))$. Its unique fixed point is E^u, and therefore one can show that E^u is in a given closed subspace of the space of continuous $E^\varphi \oplus E^s$-transverse subbundles by establishing that this closed subspace is invariant under the graph-transform.

PROPOSITION 4.2. *If φ^t is rate-symmetric then for each $H \in (0,1)$ there is a $Z > 0$ such that the graph transform preserves the space of subbundles that are H-Hölder along E^s with constant Z in local coordinates.*

Thus the graph transform preserves Hölder-continuity. More specifically, applying the graph transform to a subbundle that is H-Hölder with sufficiently large constant Z in local coordinates gives a subbundle with the same property (for the same Z and H). Thus, for any $H<1$ the unique fixed point of the graph transform lies in the space of H-Hölder subbundles, and we have shown

COROLLARY 4.3. *The strong stable and strong unstable subbundles of a rate-symmetric Anosov flow are H-Hölder for any $H<1$.*

Proof of Proposition 4.2. The advantage of the above representation is that applying the derivative amounts to simple composition. The image (in the coordinates at $\varphi^T(p)$ of the subspace is represented as the image of the linear map with matrix

$$\begin{pmatrix}\beta & 0 & 0\\ \eta & 1 & 0\\ \xi & 0 & \alpha\end{pmatrix}\begin{pmatrix}I\\ e\\ \bar e\end{pmatrix}=\begin{pmatrix}\beta\\ \eta+e\\ \xi+\alpha\bar e\end{pmatrix},$$

which is also the image of the linear map with matrix

$$\begin{pmatrix}I\\ \eta(s)\beta^{-1}+e(s)\beta^{-1}\\ \xi(s)\beta^{-1}+\alpha\bar e(s)\beta^{-1}\end{pmatrix}=:\begin{pmatrix}I\\ e(s_T)\\ \bar e(s_T)\end{pmatrix}, \tag{4}$$

where $\varphi^T(0,0,s)=:(0,0,s_T)$ in local coordinates.

We also use that by quasiconformality $df\restriction_{E^u}$ distorts volume by at most $(K\|\beta\|^*)^{\dim E^u}$, and $df\restriction_{E^s}$ distorts volume by at most $(K\|\alpha\|^*)^{\dim E^s}$, where K is as in (2). Rate-symmetry then gives

$$\|\alpha\|^*\|\beta\|^*\le 1\le K\|\alpha\|^*\cdot K\|\beta\|^* \tag{5}$$

and hence

$$\left\|\beta^{-1}\right\|=1/\|\beta\|^*\le K^2\|\alpha\|^*. \tag{6}$$

Thus,

$$\|s\|\left\|\beta^{-1}\right\|\le K^2\|\alpha\|^*\|s\|\le K^2\|\alpha s\|=K^2\|s_T\|. \tag{7}$$

To prove Proposition 4.2 take $H\in(0,1)$, then take $T>0$ large enough such that

$$\left\|\beta^{-1}\right\|=\left\|\beta_T^{-1}\right\|<K^{-2H/(1-H)} \tag{8}$$

for all p (uniform hyperbolicity), $M>0$ large enough such that $\|\eta(s)\|\le M\|s\|$ for all p (since $\eta(s)=O(s)$ uniformly), and (using that $K^{2H}\|\beta^{-1}\|^{1-H}<1$ by (8))

$$Z\ge\frac{MK^2\left(\|\alpha\|n^{1/\dim E^s}\epsilon\right)^{1-H}}{1-K^{2H}\|\beta^{-1}\|^{1-H}}\ge\sup_s\frac{MK^2\|s_T\|^{1-H}}{1-K^{2H}\|\beta^{-1}\|^{1-H}}.$$

We will use that this implies

$$ZK^{2H}\left\|\beta^{-1}\right\|^{1-H}+MK^2\|s_T\|^{1-H}\le Z. \tag{9}$$

If $\|e(s)\| \le Z\|s\|^H$, then (7) gives

$$\begin{aligned}\|e(s_T)\| &\le \left\|e(s)\beta^{-1}\right\| + \left\|\eta(s)\beta^{-1}\right\|\\ &\le Z\|s\|^H\left\|\beta^{-1}\right\| + M\|s\|\left\|\beta^{-1}\right\|\\ &\le Z(K^2\|s_T\|)^H\left\|\beta^{-1}\right\|^{1-H} + MK^2\|s_T\|\\ &= \left(ZK^{2H}\left\|\beta^{-1}\right\|^{1-H} + MK^2\|s_T\|^{1-H}\right)\|s_T\|^H\\ &\le Z\|s_T\|^H,\end{aligned}$$

where the last step used (9).

Likewise, for $H \in (0,1]$ take $T > 0$ such that $K^2(\|\alpha\|^*)^{1-H}\|\alpha\| < 1$. If $\|\bar{e}(s)\| \le Z\|s\|^H$, then

$$\begin{aligned}\|\bar{e}(s_T)\| &= \left\|[\xi(s)+\alpha\bar{e}(s)]\beta^{-1}\right\|\\ &\le \left[o(\|s\|) + Z\|s\|^H\|\alpha\|\right]\left\|\beta^{-1}\right\|\\ &\le K^2\|\alpha\|^*\left[o(\|s\|) + Z\|s\|^H\|\alpha\|\right]\\ &\le \|\alpha s\|^H K^2(\|\alpha\|^*)^{1-H}\left[Z\|\alpha\| + o\left(\|s\|^{1-H}\right)\right]\\ &\le \|s_T\|^H K^2(\|\alpha\|^*)^{1-H}\left[Z\|\alpha\| + o\left(\|s\|^{1-H}\right)\right]\\ &= \left[ZK^2(\|\alpha\|^*)^{1-H}\|\alpha\| + o\left(\|s\|^{1-H}\right)\right]\|s_T\|^H\\ &\le Z\|s_T\|^H\end{aligned}$$

for suitable Z. □

We return to this argument later. For now we note that while we need the full force of Corollary 4.3, Proposition 4.1 shows that the last part of the preceding argument is not necessary for proving regularity of $E^u \oplus E^s$.

In order to improve Proposition 4.2 to the desired Zygmund regularity we will pursue a similar strategy.

5. ZYGMUND REGULARITY

To prove Zygmund regularity of the strong unstable subbundle we vary the strategy slightly from the one employed in Proposition 4.2. Instead of showing that the space of Z-Zygmund subbundles is preserved by the graph-transform for sufficiently large Z, we start with $E^u \oplus E^s$ itself, rather than a candidate for it, and we use that for $s > \epsilon > 0$ the Zygmund condition is vacuous, because one can just choose a suitably large constant. Starting from this, the graph transform is then used to extend the Zygmund condition to ever smaller values of s, while controlling the Zygmund constant.

PROPOSITION 5.1. *The joint subbundle $E^u \oplus E^s$ of a transversely symplectic uniformly quasiconformal Anosov flow is Zygmund-regular.*

Proof. Below we will use that $\|\eta(s)+\eta(-s)\| \le C\|s\|$ for some C that is independent of t. To see this, note that by the C^2 assumption we can take M such that

$\|\eta_1(s)-\eta_1(-s)\| \le M\|s\|^2$. Here we write η_t for the corresponding derivative of φ^t for clarity. One then gets

$$\|\eta_n(s)-\eta_n(-s)\| \le \sum_{i=0}^{n-1} \|\eta_1(s_i)+\eta_1(-s_i)\|\|\beta_i\| \le M\|s\| \sum_{i=0}^{n-1} \frac{\|s_i\|^2\|\beta_i\|}{\|s\|}.$$

This implies the claim because

$$\|s_i\|^2\|\beta_i\| = \|\alpha_i s\|^2\|\beta_i\| \le \|s\|^2\|\alpha_i\|^2\|\beta_i\| \le K^2\|\alpha_i\|\|s\|^2;$$

here, the last inequality follow from $\|\alpha_i\|\|\beta_i\| \le K^2$, which is shown similarly to (5).

By Proposition 4.1 we can represent $E^u \oplus E^s$ in linearizing coordinates as

$$\begin{pmatrix} 1 \\ e \\ \bar{e} \end{pmatrix}$$

and establish Zygmund regularity of e.

To that end pick $\epsilon_2 > \epsilon_1 > 0$ with ϵ_1/ϵ_2 sufficiently small and then $Z \in \mathbb{R}$ such that $\|e(s)+e(-s)\| \le Z\|s\|$ when $\|s\| \in (\epsilon_1,\epsilon_2)$ in *any* of these local coordinate systems. We will show that, in fact $\|e(s)+e(-s)\| \le K^2(Z+C)\|s\|$ for all s. This is accomplished by applying the graph transform to get expressions for $e(s)$ on smaller domains.

In linearizing coordinates we have $-(s_t) = (-s)_t$ and hence

$$\begin{aligned}
\|e(s_t)+e(-(s_t))\| &= \|e(s_t)+e((-s)_t)\| \\
&= \left\|[\eta(s)+\eta(-s)]\beta^{-1} + [e(s)+e(-s)]\beta^{-1}\right\| \\
&\le (\|\eta(s)+\eta(-s)\| + \|e(s)+e(-s)\|)\left\|\beta^{-1}\right\| \\
&\le (O(\|s\|) + Z\|s\|)\left\|\beta^{-1}\right\| \\
&\le K^2(Z+C)\|\alpha s\| \\
&= K^2(Z+C)\|s_t\|
\end{aligned}$$

for all $t \ge T$ for some T. Here, the last inequality used $\|s\|\|\beta^{-1}\| \le K^2\|\alpha s\|$, which follows from (6).

For this T we choose ϵ_1/ϵ_2 such that the regions $\{s \mid \|s\| \in (\epsilon_1,\epsilon_2)\}$ and $\{s \mid \|s_{-T}\| \in (\epsilon_1,\epsilon_2)\}$ overlap properly, *i.e.*, $\|s_{-T}\| = \epsilon_2 \Rightarrow \|s\| > \epsilon_1$. This produces a recursive improvement as follows: For the annulus $\{s \mid \|s\| \in (\epsilon_1,\epsilon_2)\}$ we assume the Z-Zygmund condition in *all* local charts; in particular, it is known also for $\{s \mid \|s_{-T}\| \in (\epsilon_1,\epsilon_2)\}$. Applying the graph transform for time T then gives the $K^2(Z+C)$-Zygmund condition on the image of the annulus $\{s \mid \|s_{-T}\| \in (\epsilon_1,\epsilon_2)\}$ and hence on the union of this annulus with the annulus $\{s \mid \|s\| \in (\epsilon_1,\epsilon_2)\}$; by construction this is an annulus with smaller inner radius. But we can do the same again, starting with the annulus $\{s \mid \|s_{-2T}\| \in (\epsilon_1,\epsilon_2)\}$ and applying the graph transform for time $2T$. This gives *the same* $K^2(Z+C)$-Zygmund regularity on an annulus of yet smaller inner radius, and so on.

Then this recursively establishes that e, hence A and $E^u \oplus E^s$ are $K^2(Z+C)$-Zygmund. □

6. AN OBSTRUCTION TO HIGHER REGULARITY

The regularity in Proposition 5.1 is sharp. To see this, we recognize an obstruction to higher regularity.

6.1. **The longitudinal KAM-cocycle.** Recall that (M,Ψ) denotes the manifold M together with a family of charts. Fix for now such a choice. We then write the chart representation of φ^T as $\widetilde{\varphi}^T(u,\tau,s) =: (u_T, \tau + f_T(u,\tau,s), s_T)$. Geometrically, the function f_T measures the difference between the transversal at $\varphi^T(p)$ and the φ^T-image of the transversal at p as measured using the time parameter along orbits. Since φ^T maps stable manifolds to stable manifolds and unstable manifolds to unstable manifolds, we have $f_T(0,\tau,s) = 0 = f_T(u,\tau,0)$. Thus, $(0,0,0)$ is a critical point for f_T, and the obstruction *KAM* is defined as the nontrivial part of the Hessian. The Hessian has the form $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$, and is symmetric. We consider the nontrivial block:

DEFINITION 6.1. The *longitudinal KAM-cocycle* is a map

$$KAM_T^\Psi\colon M \to E^{u*} \otimes E^{s*}, \qquad p \mapsto \frac{\partial^2 \varphi_T}{\partial s \partial u}\Big|_{(0,0,0)} \big(d\Psi_p(\cdot),\, d\Psi_p(\cdot)\big).$$

In other words, for each pair of charts (and dropping Ψ in the notation), we define a quadratic form

$$KAM_{p,T}\colon E^s(p) \times E^u(p) \to \mathbb{R} \qquad \text{by} \qquad \eta'(s) := \frac{\partial \eta(s)}{\partial s} = KAM_{p,T}(\cdot,\cdot) + o(\|s\|),$$

where $\eta(s)$ is as in (3). We will show that this defines the cohomology class of a cocycle $KAM\colon M \times \mathbb{R} \to E^{u*} \otimes E^{s*}$ independently of charts.

Sometimes we write *KAM* for $KAM_{p,T}$ and $KAMs$ for $KAM(s,\cdot)$. We now study this obstruction more carefully, showing that it is a well-defined cocycle.

LEMMA 6.2. *If φ is transversely symplectic, then KAM is an additive cocycle.*

REMARK 6.3. Note that *KAM* takes values in quadratic forms, so the proper cocycle definition involves a suitable twist:

$$KAM_{p,T+S} = \alpha(0) KAM_{\varphi^T(p),S}\,\beta(0) + KAM_{T,p} \qquad \text{for all } p \in M \text{ and } T, S \in \mathbb{R}.$$

Proof. Writing

$$D\varphi^T(0,0,s) = \begin{pmatrix} \beta & 0 & 0 \\ \eta & 1 & 0 \\ \xi & 0 & \alpha \end{pmatrix} \quad \text{at } p \quad \text{and} \quad D\varphi^S(0,0,s_T) = \begin{pmatrix} \bar\beta & 0 & 0 \\ \bar\eta & 1 & 0 \\ \bar\xi & 0 & \bar\alpha \end{pmatrix} \quad \text{at } \varphi^T(p),$$

we find that

$$D\varphi^{T+S}(0,0,s) = \begin{pmatrix} * & 0 & 0 \\ \bar\eta\beta + \eta & 1 & 0 \\ * & 0 & * \end{pmatrix}.$$

Using $\bar{\eta}(0) = 0$ this gives

$$KAM_{p,T+S} = \frac{d}{ds}(\bar{\eta}\beta + \eta)\big|_{s=0} = \alpha(0)\left(\frac{d}{ds}\bar{\eta}(0)\right)\beta(0) + \frac{d}{ds}\eta(0)$$
$$= \alpha(0) KAM_{\varphi^T(p),S}\beta(0) + KAM_{T,p}. \quad \square$$

LEMMA 6.4. *The cohomology class of the longitudinal KAM-cocycle is unaffected by coordinate changes.*

Proof. Consider coordinate changes to coordinates that also have our desired list of properties. To see how the longitudinal KAM-cocycle changes we examine the change in the differential of φ^T entailed by the coordinate change. We need only study points on the stable leaf. To do the coordinate calculations we agree that the coordinate change transforms variables $(\widetilde{u}, \widetilde{t}, \widetilde{s})$ to (u, t, s). Variables in coordinates at $\varphi^T(p)$ are marked by a subscript T. At a point $(0, 0, \widetilde{s})$ an allowed coordinate change has differential

$$\begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ * & 0 & c \end{pmatrix}$$

and the inverse in coordinates at $\varphi^T(p)$ is

$$\begin{pmatrix} a_T^{-1} & 0 & 0 \\ -b_T a_T^{-1} & 1 & 0 \\ * & 0 & c_T^{-1} \end{pmatrix},$$

with entries evaluated at $\widetilde{s}_T$. Note that $c = d\widetilde{s}/ds$. In these new coordinates the differential of φ^T at $(0, 0, \widetilde{s})$ becomes

$$\begin{pmatrix} a_T^{-1} & 0 & 0 \\ -b_T a_T^{-1} & 1 & 0 \\ * & 0 & c_T^{-1} \end{pmatrix} \begin{pmatrix} \beta & 0 & 0 \\ \eta & 1 & 0 \\ \xi & 0 & \alpha \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ b & 1 & 0 \\ * & 0 & c \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ -b_T a_T^{-1}\beta a + \eta a + b & 1 & 0 \\ * & * & c_T^{-1}\alpha c \end{pmatrix}.$$

Note that therefore $\dfrac{d\widetilde{s}_T}{d\widetilde{s}} = c_T^{-1}\alpha c$. This gives

$$\begin{aligned} \widetilde{KAM}_{p,T} &:= \frac{d}{d\widetilde{s}}\widetilde{\eta}\big|_{\widetilde{s}=0} \\ &= -\frac{d}{d\widetilde{s}} b_T(0) a_T^{-1}\beta a + \frac{d}{d\widetilde{s}}\eta(0) a + \frac{d}{d\widetilde{s}} b(0) \\ &= \frac{d}{d\widetilde{s}_T} b_T(0)\big(c_T^{-1}\alpha c,\, a_T^{-1}\beta a\big) + \frac{d}{ds}\eta(0) c_T^{-1} a + \frac{d}{d\widetilde{s}} b(0) \\ &= KAM_{p,T}(c_T^{-1}\cdot,\, a\cdot) - \frac{d}{d\widetilde{s}_T} b_T(0)\big(c_T^{-1}\alpha c,\, a_T^{-1}\beta a\big) + \frac{d}{d\widetilde{s}} b(0), \end{aligned} \tag{10}$$

which is cohomologous to *KAM*. $\square$

6.2. **Obstruction to "little Zygmund".**

PROPOSITION 6.5. *If $E^u \oplus E^s$ has modulus of continuity $o(|x\log|x||)$, in particular, if $E^u \oplus E^s$ is "little Zygmund" (Definition 1.7), then the longitudinal KAM-cocycle is null cohomologous.*

Proof. Let p be a T-periodic point. Then,

$$e(s_T)\beta = \eta(s) + e(s) = KAM_{p,T}(s,\cdot) + e(s) + o(\|s\|).$$

Recursively, this gives

$$e(s_{nT})\prod_{i=0}^{n-1}\beta(s_{iT}) = \sum_{i=0}^{n-1} KAM_{\varphi^{iT}p,T}\left(s_{iT}, \prod_{i=0}^{n-1}\beta(s_{iT})\cdot\right) + e(s) + \sum_{i=0}^{n-1} o(s_{iT})\prod_{i=0}^{n-1}\beta(s_{iT}).$$

If e has modulus of continuity $o(|x\log|x||)$, then

$$\frac{e(s_{nT})}{\|s_{nT}\|\log\|s_{nT}\|} \to 0 \qquad \text{as } n\to\infty,$$

and by rate symmetry, so does

$$\frac{e(s_{nT})\prod_{i=0}^{n-1}\beta(s_{iT})}{\|s\|\log\|s_{nT}\|} = \frac{\sum_{i=0}^{n-1} KAM_{\varphi^{iT}p,T}\left(s_{iT}, \prod_{i=0}^{n-1}\beta(s_{iT})\cdot\right)}{\|s\|\log\|s_{nT}\|} + \frac{e(s)}{\|s\|\log\|s_{nT}\|} + o(n)$$

and hence also

$$\frac{1}{n}\sum_{i=0}^{n-1} KAM_{\varphi^{iT}p,T}\left(s_{iT}, \prod_{i=0}^{n-1}\beta(s_{iT})\cdot\right) = \frac{1}{n}\sum_{i=0}^{n-1} KAM_{p,T}\left(s_{iT}, \prod_{i=0}^{n-1}\beta(s_{iT})\cdot\right).$$

Thus, $KAM_{p,T} = KAM = 0$ by rate symmetry. □

7. LIPSCHITZ CONTINUITY

We now study what happens when the longitudinal KAM-cocycle is trivial. As a first step we show that this allows more perfectly adapted coordinate systems.

LEMMA 7.1. *If the longitudinal KAM-cocycle is a coboundary then there are nonstationary local coordinates in which it vanishes identically.*

Proof. If the longitudinal KAM-cocycle is null-cohomologous then there is a bilinear form k such that $KAM_{p,T} = k_{\varphi^T(p)}(\alpha\cdot,\beta\cdot) - k_p$ for all $p\in M$, $T\in\mathbb{R}$. With the notations from the proof of Lemma 6.4, define a coordinate change at p by

$$\begin{pmatrix} u\\ t\\ s\end{pmatrix} = \begin{pmatrix} \widetilde{u}\\ \widetilde{t} + k_p(\widetilde{u},\widetilde{s})\\ \widetilde{s}\end{pmatrix},$$

whose differential at $(0,0,\widetilde{s})$ is

$$\begin{pmatrix} I & 0 & 0\\ k_p\widetilde{s} & 1 & 0\\ 0 & 0 & I\end{pmatrix}.$$

Then, (10) with $a = c = a_T^{-1} = c_T^{-1} = I$ gives

$$\begin{aligned}\widetilde{KAM}_{p,T} &= KAM_{p,T} - \frac{d}{d\tilde{s}_T} b_T(0)(\alpha\,\cdot, \beta\,\cdot) + \frac{d}{d\tilde{s}} b(0)\\ &= KAM_{p,T} - k_{\varphi^T(p)}(\alpha\,\cdot, \beta\,\cdot) + k_p\\ &= 0.\end{aligned}$$

□

Proving that $E^u \oplus E^s$ is Lipschitz continuous when the KAM-cocycle vanishes involves estimates similar to those in the proof of Proposition 4.2, but the reasoning follows the lines of Proposition 5.1.

PROPOSITION 7.2. *If φ is a uniformly quasiconformal transversely symplectic Anosov flow with $KAM = 0$ then $E^u \oplus E^s$ is Lipschitz continuous.*

Proof. We will use that in our new coordinates the cocycle is trivial, *i.e.,* $\eta'(0) = 0$ in (4), so $\|\eta_T(s)\| \le C\|s\|^2$ for some C that is uniform in s *and* T. One sees this by recursively obtaining $\|\eta_n(s)\| \le C'\big[\sum_{i=0}^{\infty} \|\alpha(s_i)\| K^{2i}\big]\|s\|^2$ from $\|\eta_1(s)\| \le C'\|s\|$. (Compare with the beginning of the proof of Proposition 5.1.)

Analogously to Proposition 5.1 pick $\epsilon_2 > \epsilon_1 > 0$ and Z such that $\|e(s)\| \le Z\|s\|$ for $\|s\| \in (\epsilon_1, \epsilon_2)$ in any coordinate system. Then the first calculation in Proposition 4.2 becomes

$$\begin{aligned}\|e(s_T)\| = \|\eta_T(s) + e(s)\| \left\|\beta_T^{-1}(s)\right\| &\le (C\|s\|^2 + Z\|s\|) \left\|\beta_T^{-1}(s)\right\|\\ &\le (Z + C\|s\|) \left\|\beta^{-1}\right\| \|s\|\\ &\le Z(1 + C\|s\|/Z)K^2 |s_T|.\end{aligned}$$

From here one concludes as in Proposition 5.1. □

8. RIGIDITY

In this section we provide the proof of Theorem 1.2. By Theorem 1.12, the flow φ^λ is up to finite covers C^∞ orbit equivalent either to the suspension of a hyperbolic automorphism of a torus, or to the geodesic flow of a closed hyperbolic manifold. Since the unit sphere bundle SN is not diffeomorphic to the suspension manifold of a torus, we have

LEMMA 8.1. *With the notations of Theorem 1.2, the flow φ^λ is C^∞ orbit equivalent to the geodesic flow ψ of a closed hyperbolic manifold.*

Using the C^∞ orbit equivalence, we can suppose that ψ is also defined on SN. By construction, ψ and φ^λ have the same oriented orbits.

LEMMA 8.2. *The magnetic 2-form $\lambda\Omega$ is exact.*

Proof. We first show that ω_λ is ψ-invariant. Since ψ is a C^∞ time change of φ^λ, there exists a positive C^∞ function f on SN such that ψ is generated by X/f, where X denotes the vector field that generates φ^λ. Now, ω_λ is closed and $i_X\omega_\lambda = 0$, so

$$\mathscr{L}_{X/f}\omega_\lambda = d\, i_{X/f}\omega_\lambda + i_{X/f} d\omega_\lambda = 0,$$

and ω_λ is ψ-invariant.

We now show that the 2-form $\lambda\pi^*\Omega$ of SN is exact.

The canonical 1-form A for ψ is C^∞ and ψ-invariant, hence so is the 2-form dA.

If $\dim N \geq 4$, the space of smooth ψ-invariant 2-forms of SN is 1-dimensional [16], so there exists $a \in \mathbb{R}$ such that $\omega_\lambda = a \cdot dA$, *i.e.,*

$$d\alpha - \lambda\pi^*\Omega = a \cdot dA,$$

where $\pi\colon SN \to N$ denotes the canonical projection. Thus, $\lambda\pi^*\Omega$ is exact.

If $\dim N = 3$, the space of smooth ψ-invariant 2-forms is 2-dimensional [16], and $A \wedge \omega_\lambda \wedge \omega_\lambda$ is a C^∞ volume form of SN. So the transverse symplectic form ω_λ is stable in the terminology of [22], which implies that $\lambda\pi^*\Omega$ is exact by [22, Theorem A]. Therefore, $\lambda\pi^*\Omega$ of SN is exact also in this case.

The Gysin sequence of $\pi\colon SN \to N$ shows that $\pi^*\colon H^2(N,\mathbb{R}) \to H^2(SN,\mathbb{R})$ is an isomorphism for $n \geq 4$ and injective for $n = 3$. Therefore, since $\lambda\pi^*\Omega$ is exact, so is $\lambda\Omega$. □

LEMMA 8.3. $\lambda\Omega = 0$.

Proof. By the previous lemma, there is a C^∞ 1-form θ on N such that $\lambda\Omega = \lambda \cdot d\theta$, and it suffices to show that $\lambda\theta$ is closed.

For $\lambda \in \mathbb{R}$ small, define the Finsler metric $F_\lambda := \sqrt{g} - \lambda\theta$. Let φ_F^λ be the Finsler geodesic flow of F_λ defined on the Finsler unit sphere bundle $S_{F_\lambda}N = \{v \in TN \mid F_\lambda(v) = 1\}$. Then φ^λ and φ_F^λ are C^∞ orbit equivalent [21].

Since φ^λ is uniformly quasiconformal, φ_F^λ is also uniformly quasiconformal, which implies in particular that the weak-stable and weak-unstable subbundles of φ_F^λ are both C^∞ [4, 23]. Since in addition φ_F^λ is of contact type, the strong-stable and strong-unstable subbundles of φ_F^λ are also C^∞. The claim is now a consequence of the following result:

THEOREM 8.4 ([5, Corollary 6])**.** *For $\lambda \in \mathbb{R}$ small, if the geodesic flow of the negatively curved Finsler metric $\sqrt{g} - \lambda\theta$ has C^2 strong-stable and strong-unstable subbundles, then $\lambda\theta$ is closed.* □

Since $\lambda\Omega \equiv 0$, φ^λ is the geodesic flow of the negatively curved Riemannian metric g. Since in addition φ^λ is uniformly quasiconformal, the principal result of [24] implies that g has constant negative curvature.

APPENDIX A. THE ZYGMUND CONDITION IN CHARTS

In Definition 1.7 we introduced the notion of Zygmund-regularity and said that for maps between manifolds this definition is applied in smooth local coordinates. Implicit in this assertion is that this notion is unaffected by precomposing with a smooth map. While this is widely used in dynamical systems, no proof seems to be readily available in the literature, possibly because this notion was introduced in connection with Fourier series, where this is not a natural question to ask.

We here provide a proof of this fact that was kindly supplied to us by Rafael de la Llave.

THEOREM A.1. *If X, Y, Z are normed linear spaces, $U \subset X$ and $V \subset Y$ are both open, $f\colon V \to Z$ is Zygmund-regular, and $g\colon U \to V$ is $C^{1+\alpha}$ for some $\alpha \in (0,1)$ then $f \circ g$ is also Zygmund-regular.*

REMARK A.2. The more natural result would be that a composition of Zygmund-regular maps is itself Zygmund-regular, but this is not true (*e.g.*, for $f(x) = g(x) = x\log|x|$). It is not even clear whether $f \circ g$ is Zygmund-regular if g is only C^1. Since the $C^{1+\alpha}$ hypothesis is pervasive in hyperbolic dynamical systems, the theorem we supply here is useful enough.

Proof of Theorem A.1. Suppose $\|f(x+h)+f(x-h)-2f(x)\| \le Z\|h\|$ for all $x \in V$ and sufficiently small $\|h\|$ and write

$$\begin{aligned}&(f\circ g)(x+h)+(f\circ g)(x-h)-2(f\circ g)(x)\\ &\qquad = f[g(x)+Dg(x)h+R_x(h)]+f[g(x)-Dg(x)h+R_x(-h)]-2(f\circ g)(x),\end{aligned}$$

where $\|R_x(\pm h)\| \le O\big(\|h\|^{1+\alpha}\big)$ because $g \in C^{1+\alpha}$.

Since Zygmund regularity implies modulus of continuity $O(|x\log|x||)$ [25, Chapter II, Theorem (3·4)], we also have

$$\|f[g(x)\pm Dg(x)h+R_x(\pm h)]-f[g(x)\pm Dg(x)h]\| \le O\big(\|h\|^{1+\alpha}\big|\log\|h\|^{1+\alpha}\big|\big).$$

Consequently,

$$\begin{aligned}\|(f\circ g)(x+h)+&(f\circ g)(x-h)-2(f\circ g)(x)\|\\ &\le \|f[g(x)+Dg(x)h]+f[g(x)-Dg(x)h]-2f(g(x))\|\\ &\qquad + O\big(\|h\|^{1+\alpha}\big|\log\|h\|^{1+\alpha}\big|\big)\\ &\le Z\|Dg(x)h\| + O\big(\|h\|^{1+\alpha}\big|\log\|h\|^{1+\alpha}\big|\big)\\ &\le Z'\|h\|\end{aligned}$$

for some $Z' > 0$ and sufficiently small $\|h\|$. □

REMARK A.3. The same argument shows also that the property of being "little Zygmund" is preserved by smooth changes of variable.

Although this is not needed in the present work, we also provide a proof (supplied by Rafael de la Llave as well) that the property of being $C^{1+\text{Zygmund}}$ (that is, of being differentiable with Zygmund derivative) is preserved under smooth coordinate changes. This is interesting in part because one needs less regularity in the coordinate change than $C^{2+\alpha}$, which is what analogy with the preceding result would suggest. Also, $C^{1+\text{Zygmund}}$ is the critical regularity in [15].

THEOREM A.4. *If X, Y, Z are normed linear spaces, $U \subset X$ and $V \subset Y$ are both open, $f\colon V \to Z$ is $C^{1+\text{Zygmund}}$, and $g\colon U \to V$ is $C^{1+\text{Lipschitz}}$ then $f \circ g$ is also $C^{1+\text{Zygmund}}$.*

Proof. Given that Df is Zygmund and Dg is Lipschitz we need to establish that $D(f\circ g) = ((Df)\circ g)Dg$ is Zygmund, which follows from the next lemma. □

LEMMA A.5. *If f is Zygmund and g is Lipschitz, then $f \cdot g$ is Zygmund.*

Proof. Write

$$\begin{aligned}
&\|f(x+h)g(x+h)+f(x-h)g(x-h)-2f(x)g(x)\| \\
&\qquad\leq \|f(x+h)+f(x-h)-2f(x)\|\|g(x)\| \\
&\qquad\quad+\|f(x+h)\|\|g(x+h)-g(x)\| \\
&\qquad\quad+\|f(x-h)\|\|g(x-h)-g(x)\|
\end{aligned}$$

and note that each summand is $O(h)$. □

REMARK A.6 (de la Llave)**.** It seems that for $k \geq 2$ the composition of $C^{k+\text{Zygmund}}$ maps is again $C^{k+\text{Zygmund}}$. For instance, if $k=2$ we find that

$$D^2(f\circ g)=(D^2 f)\circ g\, Dg\, Dg+(Df\circ g)D^2 g$$

is Zygmund because $(D^2 f)\circ g$ and $D^2 g$ are Zygmund, and Dg and $Df\circ g$ are $C^{1+\text{Zygmund}}$, hence Lipschitz.

For Zygmund-regularity of fractional order these issues are treated comprehensively in [19].

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YONG FANG <yfang@u-cergy.fr>: Département de Mathématiques, Université de Cergy-Pontoise, avenue Adolphe Chauvin, 95302, Cergy-Pontoise Cedex, France

PATRICK FOULON <foulon@math.u-strasbg.fr>: Institut de Recherche Mathematique Avancée, UMR 7501 du Centre National de la Recherche Scientifique, 7 Rue René Descartes, 67084 Strasbourg Cedex, France

BORIS HASSELBLATT <Boris.Hasselblatt@tufts.edu>: Department of Mathematics, Tufts University, Medford, MA 02155, USA