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LONGITUDINAL FOLIATION RIGIDITY AND LIPSCHITZ-CONTINUOUS INVARIANT FORMS FOR HYPERBOLIC FLOWS

YONG FANG, PATRICK FOULON, AND BORIS HASSELBLATT

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Abstract. In several contexts the defining invariant structures of a hyperbolic dynamical system are smooth only in systems of algebraic origin, and we prove new results of this smooth rigidity type for a class of flows.

For a transversely symplectic uniformly quasiconformal C^2 Anosov flow on a compact Riemannian manifold we define the *longitudinal KAM-cocycle* and use it to prove a rigidity result: The joint stable/unstable subbundle is Zygmund-regular, and higher regularity implies vanishing of the KAM-cocycle, which in turn implies that the subbundle is Lipschitz-continuous and indeed that the flow is smoothly conjugate to an algebraic one. To establish the latter, we prove results for algebraic Anosov systems that imply smoothness and a special structure for any Lipschitz-continuous invariant 1-form.

We obtain a pertinent geometric rigidity result: Uniformly quasiconformal magnetic flows are geodesic flows of hyperbolic metrics.

Several features of the reasoning are interesting: The use of exterior calculus for Lipschitz-continuous forms, that the arguments for geodesic flows and infranilmanifoldautomorphisms are quite different, and the need for mixing as opposed to ergodicity in the latter case.

1. Introduction

1.1. **Statement of main result.** Anosov systems (1.5), both diffeomorphisms and flows, exhibit interesting phenomena of smooth and geometric rigidity in connection with the degree of (transverse) regularity of the (weak) stable and unstable subbundles of these systems. The seminal result was the study of volume-preserving Anosov flows on 3-manifolds by Hurder and Katok [13], which showed that the weak-stable and weak-unstable foliations are $C^{1+\text{Zygmund}}$ and that there is an obstruction to higher regularity whose vanishing implies smoothness of these foliations. This, in turn, happens only if the Anosov flow is smoothly conjugate to an algebraic one. The cocycle obstruction described by Katok and Hurder was first observed by Anosov and is the first nonlinear coefficient in the Moser normal form. Therefore one might call it the KAM-cocycle. This should not be confused with "KAM" as in "Kolmogorov–Arnold–Moser", and Hurder and Katok refer to this object as the *Anosov-cocycle*.

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In [7] we showed some analogous rigidity features associated with the *longitudinal* direction, *i.e.,* associated with various degrees of regularity of the sum of the *strong* stable and unstable subbundles: For a volume-preserving Anosov flow on a 3-manifold the strong stable and unstable foliations are Zygmund-regular [16, Section II.3, (3·1)], see 1.6, and there is an obstruction to higher regularity, which admits a direct geometric interpretation and whose vanishing implies high smoothness of the joint strong subbundle and that the flow is either a suspension or a contact flow. In the papers announced here [8, 9] we push this to higher-dimensional systems:

Theorem 1.1. *Let M be a compact Riemannian manifold of dimension at least 5, $k \geq 2$, $\varphi\colon \mathbb{R} \times M \to M$ a uniformly quasiconformal (1.8) transversely symplectic C^k Anosov flow.*

Then $E^u \oplus E^s$ is Zygmund-regular and there is an obstruction to higher regularity that defines the cohomology class of a cocycle we call the longitudinal KAM-cocycle*. This obstruction can be described geometrically as the curvature of the image of a transversal under a return map, and the following are equivalent:*

1. *$E^u \oplus E^s$ is "little Zygmund" (see 1.6);*
2. *The longitudinal KAM-cocycle is a coboundary;*
3. *$E^u \oplus E^s$ is Lipschitz-continuous;*
4. *φ is up to finite covers, constant rescaling and a* canonical time-change *(1.5) C^k-conjugate to the suspension of a symplectic Anosov automorphism of a torus or the geodesic flow of a real hyperbolic manifold.*

To show that (3) implies (4) we study the *canonical 1-form* (1.5) of the time-change of a geodesic flow or of the suspension of an infranilmanifoldautomorphism, and because we only have Lipschitz-continuity at our disposal, we need to explore how smooth-rigidity results can be pushed to the lowest conceivable regularity. This requires two main results. On the one hand, a Lipschitz-continuous 1-form whose exterior derivative is invariant under the geodesic flow of a negatively curved locally symmetric space must be (a constant multiple of) the canonical 1-form for the flow. On the other hand, an essentially bounded 2-form invariant under an infranilmanifoldautomorphism is smooth. A special case of the second result is that in which the 2-form arises as the exterior derivative of a Lipschitz-continuous 1-form, in which case it vanishes. Thus, both results involve exterior calculus of Lipschitz-continuous 1-forms. It is clear that this has to be done with care, and we invoke results to the effect that, for instance, the classical Stokes Theorem holds for Lipschitz-continuous forms [4].

Given this common motivation, it is surprising that our two separate results involve rather different arguments for geodesic flows on the one hand and suspensions on the other hand.

A particular point of interest is in this respect that while, like with many other results in hyperbolic dynamics, ergodic theory enters the proof in the case of geodesic flows only to the extent that we use ergodicity of the geodesic flow, it turns out that for the case of a suspension we use in an essential way that the infranilmanifoldautomorphism is indeed mixing rather than merely ergodic. This reflects the need to deal with parabolic effects due to the nilpotent part.

Just as 1.1 concludes with a stronger rigidity statement than its 3-dimensional precursor 1.7 below, we are able to prove a geometric rigidity result that is stronger than the 3-dimensional version deduced from 1.7 in [3]:

Theorem 1.2. *Let (N, g) be a n-dimensional closed negatively curved Riemannian manifold and Ω a C^∞ closed 2-form of N. For small $\lambda \in \mathbb{R}$, let φ^λ be the magnetic Anosov flow of the pair $(g, \lambda\Omega)$. Suppose that $n \geq 3$ and φ^λ is uniformly quasi-conformal. Then g has constant negative curvature and $\lambda\Omega = 0$. In particular, the longitudinal KAM-cocycle of φ^λ is a coboundary.*

Magnetic flows are introduced below; for the moment it may suffice to remark that the magnetic flow for $\lambda = 0$ is the geodesic flow.

1.2. **Background and terminology.** We now introduce the notions that play a role in this result and the proof.

Definition 1.3. Let φ be a C^∞ flow on a closed manifold M. Denote by X the generating vector field of φ. The flow φ is said to be *transversely symplectic* if there exists a C^∞ closed 2-form ω on M such that $\mathrm{Ker}\omega = \mathbb{R}X$. The closed 2-form ω is said to be the *transverse symplectic form* of φ. It is easy to see that ω is φ-invariant.

Examples of transversely symplectic flows can be obtained via the following classical construction: Let (N, ω^N) be a C^∞ symplectic manifold and $H\colon N \to \mathbb{R}$ a C^∞ Hamiltonian function. The Hamiltonian flow φ^N of H with respect to ω^N is generated by the vector field X^N determined by $i_{X^N}\omega^N = -dH$. Observe that φ^N preserves all the energy levels of H. Therefore, if the function H is proper and $a \in \mathbb{R}$ is a regular value of H, then the energy level $H^{-1}(a)$ is a closed φ^N-invariant C^∞ submanifold of N. Let $M = H^{-1}(a)$ and $\varphi = \varphi^N |_M$. It is easy to verify that φ is transversely symplectic with respect to the closed 2-form $\omega^N |_M$.

Magnetic flows are important examples of transversely symplectic flows and are constructed as follows:

Definition 1.4. Let (N, g) be a closed C^∞ Riemannian manifold and let Ω be a C^∞ closed 2-form of N. Let α denote the C^∞ 1-form on TN obtained by pulling back the Liouville 1-form of T^*N via the Riemannian metric. For $\lambda \in \mathbb{R}$, the *twisted symplectic structure* ω_λ is defined as

$$\omega_\lambda = d\alpha - \lambda\pi^*\Omega,$$

where $\pi\colon TN \to N$ denotes the canonical projection. Let $H\colon TN \to \mathbb{R}$ be the Hamiltonian function defined as

$$H(v) = \frac{1}{2}g(v, v)$$

for any $v \in TN$. The energy level $H^{-1}(1/2)$ is the unit sphere bundle SN. Let φ^λ be the restriction to SN of the Hamiltonian flow of H with respect to ω_λ. It is clear that φ^λ is a transversely symplectic flow with respect to $\omega_\lambda |_{SN}$, which is said to be the *magnetic flow* of the pair $(g, \lambda\Omega)$.

If the Riemannian metric g is negatively curved and λ is small enough, then the magnetic flow φ^λ is a transversely symplectic Anosov flow.

Definition 1.5 ([14])**.** An Anosov flow on a manifold M is a smooth flow φ^t with

- an invariant decomposition $TM = X \oplus E^u \oplus E^s$ (where $X = \dot\varphi \neq 0$ is the generator of the flow and E^u and E^s are called the unstable and stable subbundles) and

- a Riemannian metric on M such that $D\varphi^t\restriction_{E^s}$ and $D\varphi^{-t}\restriction_{E^u}$ are contractions whenever $t > 0$.

The definition of Anosov diffeomorphism is analogous with $t \in \mathbb{Z}$ and X absent.

The *canonical 1-form* A of an Anosov flow φ^t is defined by the conditions $A(X) = 1$ and $E^u, E^s \subset \ker A$. A *canonical time-change* is defined using a closed 1-form α by replacing the generator X of the flow by the vector field $X/(1 + \alpha(X))$, provided α is such that the denominator is positive. (See 2 for more on canonical time-changes.)

The subbundles are invariant and (Hölder-) continuous with smooth integral manifolds W^u and W^s that are coherent in that $q \in W^u(p) \Rightarrow W^u(q) = W^u(p)$. W^u and W^s define laminations (continuous foliations with smooth leaves).

Definition 1.6. A function f between metric spaces is said to be *Hölder continuous* if there is an $H > 0$, called the Hölder exponent, such that $d(f(x), f(y)) \le \text{const} \cdot d(x,y)^H$ whenever $d(x,y)$ is sufficiently small. We specify the exponent by saying that a function is H-Hölder. A continuous function $f\colon U \to L$ on an open set $U \subset L'$ in a normed linear space to a normed linear space is said to be *Zygmund-regular* if there is $Z > 0$ such that $\|f(x+h) + f(x-h) - 2f(x)\| \le Z\|h\|$ for all $x \in U$ and sufficiently small $\|h\|$. To specify a value of Z we may refer to a function as being Z-Zygmund. The function is said to be *"little Zygmund"* (or "zygmund") if $\|f(x+h) + f(x-h) - 2f(x)\| = o(\|h\|)$. For maps between manifolds these definitions are applied in smooth local coordinates.

Zygmund regularity implies modulus of continuity $O(|x \log |x||)$ and hence H-Hölder continuity for all $H < 1$ [16, Section II.3, Theorem (3·4)]. It follows from Lipschitz-continuity and hence from differentiability. Being "little Zygmund" implies having modulus of continuity $o(|x \log |x||)$ and follows from differentiability but not from Lipschitz-continuity.

The regularity of the unstable subbundle E^u is usually substantially lower than that of the weak-unstable subbundle $E^u \oplus E^\varphi$. The exception are geodesic flows, where the strong unstable subbundle is obtained from the weak-unstable subbundle by intersecting with the kernel of the invariant contact form. This has the effect that the strong-unstable and weak-unstable subbundles have the same regularity. However, time changes affect the regularity of the strong-unstable subbundle, and this is what typically keeps its regularity below C^1. In [7] we presented a *longitudinal KAM-cocycle* that is the obstruction to differentiability, and we derived higher regularity from its vanishing.

Theorem 1.7 ([7, Theorem 3])**.** *Let M be a 3-manifold, $k \ge 2$, $\varphi\colon \mathbb{R} \times M \to M$ a C^k volume-preserving Anosov flow. Then $E^u \oplus E^s$ is Zygmund-regular, and there is an obstruction to higher regularity that can be described geometrically as the curvature of the image of a transversal under a return map. This obstruction defines the cohomology class of a cocycle (the longitudinal KAM-cocycle), and the following are equivalent:*

(1) *$E^u \oplus E^s$ is "little Zygmund" (see 1.6).*
(2) *The longitudinal KAM-cocycle is a coboundary.*
(3) *$E^u \oplus E^s$ is Lipschitz-continuous.*
(4) *$E^u \oplus E^s \in C^{k-1}$.*
(5) *φ is a suspension or contact flow.*

In (5) no stronger rigidity should be expected because $E^u \oplus E^s$ is smooth for all suspensions and contact flows. See [15, 3] for applications of this to magnetic flows.

The work by Hurder and Katok in [13] inspired developments of substantial extensions to higher dimensions, see, for example, [12]. The present work extends our earlier work to higher-dimensional systems in this "longitudinal" context. This requires somewhat stringent assumptions, however.

Definition 1.8. An Anosov flow is said to be *uniformly quasiconformal* if

$$K_i(x,t) := \frac{\|d\varphi^t\restriction_{E^i}\|}{\|d\varphi^t\restriction_{E^i}\|^*} \tag{1}$$

is bounded on $\{u,s\} \times M \times \mathbb{R}$, where $\|A\|^* := \min_{\|v\|=1} \|Av\|$ is the *conorm* of a linear map A.

1.3. **Rigidity.** The proof that (1)⇒(2)⇒(3) in 1.1 largely follows the line of reasoning already presented in [7] and appears in [8].

In the 3-dimensional case we showed that smoothness of $E^u \oplus E^s$ implies that φ is a suspension or contact flow, but in the present situation we obtain more detailed information because of the quasiconformality-assumption. This uses a rigidity theorem by Fang:

Theorem 1.9 ([5, Corollary 3])**.** *Let M be a compact Riemannian manifold and $\varphi\colon \mathbb{R} \times M \to M$ a transversely symplectic Anosov flow with $\dim E^u \geq 2$ and $\dim E^s \geq 2$. Then φ is quasiconformal if and only if φ is up to finite covers C^∞ orbit equivalent either to the suspension of a symplectic hyperbolic automorphism of a torus, or to the geodesic flow of a closed hyperbolic manifold.*

This also serves to illustrate that the assumption of uniform quasiconformality is quite restrictive. We should also point out that our result about rigidity of the situation in which $E^u \oplus E^s \in C^1$ overlaps with a closely related one by Fang, although the proof is independent:

Theorem 1.10 ([5, Corollary 2])**.** *Let φ be a C^∞ volume-preserving quasiconformal Anosov flow. If $E^s \oplus E^u \in C^1$ and $\dim E^u \geq 3$ and $\dim E^s \geq 2$ (or $\dim E^s \geq 3$ and $\dim E^u \geq 2$), then φ is up to finite covers and a constant change of time scale C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to a canonical time change (1.5) of the geodesic flow of a closed hyperbolic manifold.*

1.9 yields "(3)⇒(4)" in 1.1 due to the following results.

Theorem 1.11. *Let M be a compact locally symmetric space with negative sectional curvature and consider a time-change of the geodesic flow whose canonical 1-form is Lipschitz-continuous. Then the canonical form of the time-change is C^∞, and the time-change is a canonical time-change.*

Theorem 1.12. *Let ψ be a hyperbolic automorphism of a torus or a nilmanifold $\Gamma\backslash M$ and consider a time-change of the suspension whose canonical 1-form is Lipschitz-continuous. Then the canonical form of the time-change is C^∞, and the time-change is a canonical time-change.*

Indeed, by 1.9, φ is smoothly orbit equivalent to either the geodesic flow of a real hyperbolic manifold or a suspension of a symplectic automorphism of an n-torus. To show that the flow is, after rescaling, smoothly *conjugate* to one of these models, use the orbit equivalence to regard the canonical form for φ as an invariant form for the algebraic system and then apply 1.11 or 1.12.

After introducing some background on canonical time-changes, we outline how to establish 1.11 and 1.12.

2. Canonical time-changes

We make a few remarks here about canonical time-changes (1.5) because these are not frequently encountered in the literature; they may serve to show why this is a natural class of time-changes to expect in rigidity results for flows.

Proposition 2.1 (Trivial time-changes)**.** *Consider a flow φ^t generated by the vector field X and a smooth function $f\colon M \to \mathbb{R}$ such that $1 + df(X) > 0$. Then $\Psi\colon x \mapsto \varphi^{f(x)}(x)$ conjugates the flow generated by the vector field $X_f := \dfrac{X}{1+df(X)}$ to φ^t.*

Proof. Smoothness of f and $1+df(X) > 0$ ensure that Ψ is a diffeomorphism. Now we write $x_t = \varphi^t(x)$ and use the chain rule to compute

$$\begin{aligned} d\Psi(X(x)) &= \frac{d\Psi}{dt} = \frac{d}{dt}\varphi^{f(x_t)}(x_t)_{|t=0} = \frac{d\varphi}{dt}_{|t=0} df(X)(x) + X(\varphi^{f(x)}(x)) \\ &= X(\varphi^{f(x)}(x)) \cdot df(X)(x) + X(\varphi^{f(x)}(x)) = (1 + df(X)(x))X(\varphi^{f(x)}(x)), \end{aligned}$$

which gives $d\Psi(X_f) = X$ upon division by $1 + df(X)(x)$. □

Proposition 2.2 (Cohomology class)**.** *If α and β are cohomologous closed 1-forms with $1+\alpha(X) > 0$ and $1+\beta(X) > 0$ then the associated canonical time-changes of X are smoothly conjugate.*

Remark 2.3. This tells us that the cohomology class of α is the material ingredient in a canonical time-change by α.

Proof. Writing $\beta = \alpha + df$ with smooth f we observe that

$$\frac{X}{1+\beta(X)} = \frac{X}{1+\alpha(X)+df(X)} = \frac{\frac{X}{1+\alpha(X)}}{1+df(\frac{X}{1+\alpha(X)})} = \Big(\frac{X}{1+\alpha(X)}\Big)_f.$$

Now use 2.1. □

Proposition 2.4 (Regularity)**.** *Suppose X_0 generates an Anosov flow, and that α is a closed 1-form such that $1+\alpha(X_0) > 0$. If A_0 denotes the canonical form for X_0 then $A := A_0 + \alpha$ is the canonical form for $X := \frac{X_0}{1+\alpha(X_0)}$.*

Remark 2.5. This shows, in particular, that canonical time-changes with smooth closed forms do not affect the regularity of the canonical form.

Proof. We first note that two invariant 1-forms for an Anosov flow are proportional: Both being constant on X, this follows from the fact that a continuous 1-form that vanishes on X is trivial [6, Lemma 1].

Since α is closed we have $dA = dA_0$. Also

$$A(X) = \frac{A_0(X_0) + \alpha(X_0)}{1+\alpha(X_0)} = 1,$$

which implies that $\mathcal{L}_X A = 0$, *i.e.,* A is X-invariant and hence proportional to the canonical 1-form of X. But $A(X) = 1$ then implies that A is equal to the canonical 1-form of X. □

3. Rigidity results

The results that imply 1.11 and 1.12 are of independent interest and will therefore appear in a separate publication from the application [9] to quasiconformal Anosov flows.

Theorem 3.1. *Let M be a compact locally symmetric space with negative sectional curvature and suppose A is a Lipschitz-continuous 1-form such that dA is invariant under the geodesic flow. Then A is C^∞, and indeed dA is a constant multiple of the exterior derivative of the canonical 1-form for the geodesic flow.*

Remark 3.2. Note that the Lipschitz assumption ensures that dA is defined almost everywhere and essentially bounded [10]. This is all we use. For comparison, we state an earlier result of Hamenstädt:

Theorem 3.3 ([11, Theorem A.3])**.** *If the Anosov splitting of the geodesic flow of a compact negatively curved manifold is C^1 and A is a C^1 1-form such that dA is invariant, then dA is proportional to the exterior derivative of the canonical 1-form of the geodesic flow.*

Proof of 1.11 from 3.1. The hypotheses of 1.11 and of 3.1 imply smoothness; we need to show that the vector field X that generates the time-change agrees with a canonical time-change of a constantly scaled version of X_0, where X_0 generates the geodesic flow. Rescale X_0 to X_0/κ, where $\kappa \in \mathbb{R}$ is defined by $dA = \kappa dB$ and then apply the canonical time-change defined by the 1-form $\bar{\alpha} := A - \kappa B$. Since the resulting vector field $\dfrac{X_0}{\kappa + \bar{\alpha}(X_0)}$ is a scalar multiple of X, the claim follows from

$$A\Big(\frac{X_0}{\kappa + \bar{\alpha}(X_0)}\Big) = \frac{A(X_0)}{\kappa B(X_0) + (A - \kappa B)(X_0)} = 1,$$

where we used $B(X_0) = 1$. That the last denominator is $A(X_0)$ and hence positive justifies the use of $\bar{\alpha}$ to define a canonical time-change. □

Theorem 3.4. *Let ψ be a hyperbolic automorphism of a torus or an infranilmanifold $\Gamma\backslash M$. Then any essentially bounded invariant 2-form is almost everywhere equal to an M-invariant (hence smooth) closed 2-form.*

If, in addition, the form is exact, then it vanishes almost everywhere.

Remark 3.5. We point out that in this proof we use that the automorphism is mixing (rather than just ergodic). The need for this is an interesting side-light on how parabolic effects enter into our considerations.

Proof of 1.12 from 3.4. Denote by A the canonical form of the time-change and by B the canonical form of the suspension. 3.4 applied to A implies that A is smooth and closed, and hence so is $\alpha := A - B$ since B is also closed. Writing X_0 for the suspension vector field we find that the canonical time-change

$$X := \frac{X_0}{1 + \alpha(X_0)} = \frac{X_0}{A(X_0)}$$

of X_0 is the given vector field since by construction $A(X) \equiv 1$. □

4. Proof of 3.1

Using exterior calculus (carefully!) we show that A is a contact form, *i.e.,* $A \wedge \bigwedge_{i=1}^{n} dA$ is a volume, and that X_0 is in the kernel of dA. By duality, for every ξ there is a $\psi(\xi)$ such that

$$dA(\xi, \cdot) = dB(\psi(\xi), \cdot);$$

this is defined whenever dA is, and we choose $\psi(X_0) = X_0$ and $\psi(\ker B) \subset \ker B$. We next show that $\psi \overset{\text{a.e.}}{=} \kappa \operatorname{Id} + N$, where $\kappa \in \mathbb{R} \setminus \{0\}$ and N is a nilpotent operator. The main effort is now directed at showing that $N = 0$, *i.e.,* that $dA \overset{\text{a.e.}}{=} \kappa\, dB$ (smoothness of A can then be obtained via some delicate exterior calculus). To that end we L^1-approximate N by a continuous operator, take the Birkhoff average of this approximation (which does not change the L^1-distance to N) and show that it is defined and continuous everywhere and so intertwined with the flow that one can apply arguments from [1, 2] to conclude that it vanishes identically. Thus N is arbitrarily L^1-close to 0 and hence vanishes itself.

Taking Birkhoff averages of the continuous L^1-approximation F of N requires substantial technical underpinnings because the Birkhoff Ergodic Theorem applies to scalar functions. Therefore we show that we can choose a measurable orthonormal frame field for E^u on M that consists of vector fields ξ chosen in such a way that

- ξ is continuous and $\mathcal{D}$-parallel along unstable leaves that are homeomorphic to Euclidean space, *i.e.,* nonperiodic leaves,
- $\xi \in E^j$ for $j = 1$ or $j = 2$,
- if $\xi \in E^j$ then the Lie bracket with the generator X of the geodesic flow is $[X, \xi] = \mathcal{D}_X \xi + j\xi = j\xi$ (the last equality uses that ξ is $\mathcal{D}$-parallel) and hence $\gamma^t(\xi) = e^{jt}\xi$.

Here $\mathcal{D}$ is a Kanai connection constructed for this purpose (as in [1, 2]), and its properties produce such a frame field. We use that the geodesic flow of a locally symmetric space admits an invariant splitting of the unstable subbundle into fast- and slow-unstable bundles corresponding to the exponents 1 and 2; we write $E^u = E^1 \oplus E^2$. In the constant-curvature case we have $E^2 = 0$.

We next show that the Birkhoff average of F has a continuous extension $\widetilde{F}$ to the entire manifold that is parallel along stable and unstable manifolds. We then follow arguments in [1, 2] to show that this implies $\widetilde{F} = 0$ and hence $N = 0$.

5. Proof of 3.4

Consider a hyperbolic automorphism ψ of either a torus or a nilmanifold $\Gamma \backslash N$. Suppose ω is an essentially bounded 2-form such that $\psi_* \omega = \omega$.

One can write a 2-form *locally* as $\sum_{1 \leq i < j \leq n} a_{ij} dx^i \wedge dx^j$, and we will look for a way of doing so *globally* and with constant coefficients. To that end we pass to the complexification of the tangent bundle and work with a basis of N-invariant sections X_i.

At one point we choose a basis in such a way that ψ is in Jordan canonical form with respect to the dual basis consisting of the forms X_i^*, *i.e.,* $\psi = \sum_i a_{ij} X_i^*$ with $A = (a_{ij})$ in Jordan form (strictly triangular since we passed to the complexification). Now translate the basis and the dual basis by N to get invariant sections. With these choices, a Jordan block for eigenvalue λ_l for the nth iterate is of the form $A_l^n = \lambda_l^n M_{\ell, \lambda_l}(n)$ with $M_{\ell, \lambda_l}(n)$ a fixed polynomial in n.

Now denote by $\Omega(x)$ the matrix that represents ω_x with respect to the invariant frame field. Then the matrix of $\psi_*\omega$ is given by ${}^t\!A\,\Omega(x)A$ and hence the iterated relation $\omega = \psi_*^n\omega$ becomes $\Omega(\psi^n(x)) = {}^t\!A^n\Omega(x)A^n$, which is bounded (in n) for almost every x (since ω is essentially bounded). Fix such an x and decompose Ω into (not necessarily square or diagonal) blocks Ω_{ij} according to the Jordan form of A, *i.e.,* in such a way that

$$\Omega_{ij}(\psi^n(x)) = {}^t\!A_i^n\Omega_{ij}(x)A_j^n = (\lambda_i\lambda_j)^n \underbrace{{}^tM_{\ell_i,\lambda_i}(n)\,\Omega_{ij}(x)\,M_{\ell_j,\lambda_j}(n)}_{=:P_{ijx}(n)},$$

where ℓ_i and ℓ_j are the sizes of the blocks A_i and A_j, respectively. For any i, j and x, $P_{ijx}(n)$ is a matrix-valued polynomial in n, and indeed it is constant:

- If $|\lambda_i\lambda_j| \neq 1$ then $P_{ijx}(n) = 0$—otherwise $|(\lambda_i\lambda_j)^n P_{ijx}(n)|$ grows exponentially and is, in particular, unbounded.
- If $|\lambda_i\lambda_j| = 1$ then $P_{ijx}(n)$ is constant (in n)—otherwise $|(\lambda_i\lambda_j)^n P_{ijx}(n)| = |P_{ijx}(n)|$ is unbounded.

We therefore have

$${}^tM_{\ell_i,\lambda_i}(n)\Omega_{ij}(x)M_{\ell_j,\lambda_j}(n) = P_{ijx}(n) = P_{ijx}(0) = \Omega_{ij}(x),$$

so $\Omega_{ij}(\psi^n(x)) = (\lambda_i\lambda_j)^n\Omega_{ij}(x)$ for all i, j.

This shows that every entry of the matrix Ω is almost everywhere equal to an eigenfunction of ψ. Since ψ is mixing, all eigenfunctions are constant, and hence Ω is almost everywhere equal to an M-invariant (hence smooth) 2-form.

To prove the last assertion of 1.12, *i.e.,* that Ω vanishes if it is exact, we introduce a notion of averaging. Let

$$\mathrm{vol}_{\mathbb{C}} := X_1^* \wedge \cdots \wedge X_n^*$$

be the N-invariant complex-valued volume form defined by the dual basis we used before. By compactness, this gives a finite volume. Then for any p-form

$$\alpha = \sum_{i_1,\ldots,i_p} \alpha_{i_1,\ldots,i_p} X_{i_1}^* \wedge \cdots \wedge X_{i_p}^*$$

we define the *average*

$$\bar\alpha := \sum_{i_1,\ldots,i_p} \Big(\int_{\Gamma\backslash N} \alpha_{i_1,\ldots,i_p}\, d\,\mathrm{vol}_{\mathbb{C}}\Big) X_{i_1}^* \wedge \cdots \wedge X_{i_p}^*$$

and prove that $\overline{d\alpha} = d\bar\alpha$ for any 1-form α. We can write

$$\psi_*\Big(\sum_i \bar\alpha_i X_i^*\Big) = \sum_i \bar\alpha_i \psi_*(X_i^*) = \sum_i (\bar\alpha_i a_{ij}) X_j^*,$$

and since the coefficients here are constant, we obtain $d\psi_*\bar\alpha = \psi_* d\bar\alpha$.

If ω is an exact ψ-invariant 2-form with constant coefficients, then we write $\omega = d\alpha$ and note that

$$\psi_* d\bar\alpha = \psi_*\overline{d\alpha} = \psi_*\bar\omega = \psi_*\omega = \omega = \bar\omega = \overline{d\alpha} = d\bar\alpha,$$

i.e., $d(\psi_*\bar\alpha - \bar\alpha) = \psi_* d\bar\alpha - d\bar\alpha = 0$. Thus, there is an f such that $\psi_*\bar\alpha - \bar\alpha = df$ and, in particular,

$$\sum \bar\alpha_i a_{ij} - \bar\alpha_j = df(X_j).$$

Since, on the other hand, $\int_{\Gamma\backslash N} df(X_j)\, d\,\mathrm{vol}_{\mathbb{C}} = 0$ and the integrand is constant, we have $\sum \bar{\alpha}_i a_{ij} = \bar{\alpha}_j$, *i.e.,* $\bar{\alpha}$ is a ψ-invariant 1-form. But then, hyperbolicity of ψ implies that $\bar{\alpha} = 0$ (see, *e.g.,* [6, Lemma 1]) and hence $\omega = \bar{\varpi} = \overline{d\alpha} = d\bar{\alpha} = 0$.

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Département de Mathématiques, Université de Cergy-Pontoise, avenue Adolphe Chauvin, 95302, Cergy-Pontoise Cedex, France

E-mail address: yfang@u-cergy.fr

Institut de Recherche Mathematique Avancée, UMR 7501 du Centre National de la Recherche Scientifique, 7 Rue René Descartes, 67084 Strasbourg Cedex, France

E-mail address: foulon@math.u-strasbg.fr

Department of Mathematics, Tufts University, Medford, MA 02155, USA

E-mail address: Boris.Hasselblatt@tufts.edu