HYPERBOLIC DYNAMICAL SYSTEMS

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CHAPTER 1

Introduction

The aim of this survey is to outline important results in the theory of uniformly hyperbolic dynamical systems on compact spaces as well as its extension to nonuniformly hyperbolic systems, and to indicate techniques used in the development of the basic theory. Accordingly, there are comments on possible methods of proof in the earlier parts, whereas later portions give an impressionistic view of several main developments of the subject. For much of the hyperbolic theory the book [**KH**] is a useful reference; the basic core is efficiently developed in [**Yc**].

Hyperbolicity is central to several other surveys in this volume [**S-HK, S-BP, S-C, S-P, S-K, S-B**] and a few more have some common topics. (Citations of surveys in this Handbook are distinguished by the prefix "S-"; those articles are listed first in the bibliography.) Accordingly, this survey concentrates on aspects of hyperbolicity that are not discussed elsewhere in this volume. Nevertheless, in order to provide a reasonable overview some occasionally substantial overlap could not be avoided. Notations used here without being defined are adopted from [**S-HK**].

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1. Historical sketch

There are two intertwined strands of the history of hyperbolic dynamics: Geodesic flows on one hand and hyperbolic phenomena ultimately traceable to some application of dynamical systems. Geodesic flows were studied, *e.g.,* by Hadamard, Hedlund, Hopf (primarily either on surfaces or in the case of constant curvature) and Anosov–Sinai (negatively curved surfaces and higher dimensional manifolds). Other hyperbolic phenomena appear in the work of Poincaré (homoclinic tangles in celestial mechanics [**Pc**]), Perron (differential equations [**Pn1**]), Cartwright, Littlewood (relaxation oscillations in radio circuits [**C, CL, Lw**]), Levinson (the van der Pol equation, [**Lv**]) and Smale (horseshoes, [**S3, S2**]), as well as countless others in recent history. In looking back, Smale [**S5**] breaks the study of hyperbolic phenomena into three strands: Poincaré–Birkhoff, (Poincaré-) Cartwright–Littlewood–Levinson and Andronov–Pontryagin–Lefschetz–Peixoto (structural stability and topology).

a. Homoclinic tangles. Poincaré came upon hyperbolic phenomena in revising his prize memoir [**Pc**] on the three-body problem before publication. He found that homoclinic tangles (which he had initially overlooked) caused great difficulty and necessitated essentially a reversal of the main thrust of that memoir [**B-G**]. He perceived that there is a highly intricate web of invariant curves and that this situation produces dynamics of unprecedented complexity. This is often viewed as the moment chaotic dynamics was first noticed. He concluded that in all likelihood the prize problem could not be solved as posed:

To find series expansions for the motions of the bodies in the solar system that converge uniformly for all time.

b. Geodesic flows. A major class of mathematical examples motivating the development of hyperbolic dynamics is that of geodesic flows of Riemannian manifolds of negative sectional curvature. Hadamard considered (noncompact) surfaces in $\mathbb{R}^3$ of negative curvature [**Hd1**] and found that if the unbounded parts are "large" (do not pinch to arbitrarily small diameter as you go outward along them) then at any point the initial directions of bounded geodesics form a Cantor set. (Since only countably many directions give geodesics that are periodic or asymptotic to a periodic one, this proves the existence of more complicated bounded geodesics.) Hadamard was fully aware of the connection to Cantor's work and to similar sets discovered by Poincaré, and he appreciated the relation between the complicated dynamics in the two contexts. Hadamard also showed that each homotopy class (except for the "waists" of cusps) contains a unique geodesic. A classic by Duhem [**D**] seized upon this to eloquently describe the dynamics of a geodesic flow in terms of what might now be called deterministic chaos: Duhem used it to illustrate that determinism in classical mechanics does not imply any practical long-term predictability. To today's reader his description amounts to a shrewd translation of symbolic dynamics into everyday language. Indeed, several authors trace back symbolic dynamics to this paper of Hadamard. Birkhoff is among them and writes about "the symbols effectively introduced by Hadamard" [**Bh3**, p.184]. (It is an unresolved question just when symbol spaces began to be perceived as dynamical systems, rather than as a coding device.)

Geodesic flows on negatively curved surfaces were again studied in the 1920s and 1930s. For constant curvature, finite volume and finitely generated fundamental group the geodesic flow was shown to be topologically transitive [**Kb, Lb**], topologically mixing [**Hl1**], ergodic [**Ho1**], and mixing [**Hl2**]. (In the case of infinitely generated fundamental group the geodesic flow may be topologically mixing without being ergodic [**Sd**]). If the curvature is allowed to vary between two negative constants then finite volume implies topological mixing (Hedlund [**Hl3**] attributes this to [**Gt**]). Finally Hopf [**Ho2**] considered compact surfaces of nonconstant (predominantly) negative curvature and was able to show ergodicity and mixing of the Liouville measure (phase volume). This is interesting because despite the ergodicity paradigm central to statistical mechanics, the Boltzmann ergodic hypothesis (under which the time average of an observable, which is an experimentally measurable quantity, agrees with the space average, which is the corresponding quantity one can compute from theory) there was a dearth of examples of ergodic Hamiltonian systems. To this day the quintessential model for the ergodic hypothesis, the gas of hard spheres, resists attempts to prove ergodicity.

c. Picking up from Poincaré. Several mathematicians, such as Hadamard, had begun to pick up some of Poincaré's work during his lifetime. Birkhoff did so soon after Poincaré's death. He addressed issues that arose from the mathematical development of mechanics and celestial mechanics such as Poincaré's last geometric theorem and the complex dynamics necessitated by homoclinic tangles [**Bh2**, Section 9]. He was also important in the development of ergodic theory (the Poincaré Recurrence Theorem is proved in Poincaré's prize memoir [**Pc**]), notably by proving the pointwise ergodic theorem.

The work of Cartwright and Littlewood during World War II on relaxation oscillations in radar circuits [**CL, C, Lw**] consciously built on Poincaré's work. Further study of the van der Pol equation by Levinson [**Lv**] contained the first example of a structurally stable diffeomorphism with infinitely many periodic points. (Structural stability originated in 1937 [**AP**] but began to flourish only 20 years later.) This was brought to the attention of Smale. Inspired by Peixoto's work, which carried out such a program in dimension two [**Px**], Smale was after a program of studying diffeomorphisms with a view to classification [**S4**]. Until alerted by Levinson, Smale conjectured that only Morse–Smale systems (finitely many periodic points with stable and unstable sets in general position) could be structurally stable [**S1**]. He eventually extracted from Levinson's work the horseshoe [**S3, S2**]. Smale in turn was in contact with the Russian school, where Anosov systems (then C- or U-systems) had been shown to be structurally stable, and their ergodic properties were studied by way of further development of the study of geodesic flows in negative curvature.

d. Modern hyperbolic dynamics. Hopf's argument had shown roughly that Birkhoff averages of a continuous function must be constant on almost every leaf of the horocycle foliation, and, since these foliations are C^1, the averages are constant a.e. He realized that much of the argument was independent of the dimension of the manifold, but could not verify the C^1 condition in higher dimension. Anosov showed that differentiability may indeed fail in higher dimension, but that the Hopf argument can still be used because the invariant laminations have an absolute continuity property [**S-HK, A1, AS, PS2, Br3, BP**], see Subsection 2.3h. It is interesting to note that hyperbolic sets were sometimes said to constitute "a Perron situation", for example by Alekseev [**Al1**, Definition 12] (in which the Smale horseshoe makes an appearance as well). Independently, Thom (unpublished) studied hyperbolic toral automorphisms and their structural stability. (The automorphism, $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, is deplorably often called the "Arnold cat map" by physicists after [**AA**, Figure 1.17]. Since there were typewritten notes by Avez that preceded the joint book and included a similar picture that used a bat, physicists should consider the term "Avez bat map"[1]. However, "auThomorphism" is my favorite.) The initial development of the theory of hyperbolic systems in the 1960s was followed by the founding of the theory of nonuniformly hyperbolic dynamical systems in the 1970s, mostly by Pesin [**Os, Pe2**] (during which time the hyperbolic theory continued its development). One of the high points in the development of smooth dynamics is the proof by Robbin, Robinson, Mañé and Hayashi that structural stability indeed characterizes hyperbolic dynamical systems. For diffeomorphisms this was achieved in the 1980s, for flows in the 1990s. Starting in the mid-eighties the field of geometric and smooth rigidity came into being. At the same time topological and stochastic properties of attractors began to be better understood with techniques that nowadays blend ideas from hyperbolic and one-dimensional dynamics.

e. The slowness of the initial development. It is interesting that Poincaré's insights took some time to have a deep impact on dynamics, particularly given such popularization as Duhem's and the great stature Poincaré acquired in his lifetime. He was a popularly known figure and was repeatedly put forward for the Nobel prize in physics [**Is**] (before his own physics Nobel prize, Subrahmanyan Chandrasekhar is said to have rated Poincaré as the best physicist never to get a Nobel prize). Hadamard developed some of Poincaré's ideas in papers where he elegantly brought out the technical points. But he, too, appears as if he was ahead of his time, for example in considering surfaces not as objects to be studied using analytic functions. His geometric approach, inspired by Poincaré, of *analysis situs* was not immediately appreciated. And although at almost any time in the 20th century there was progress on some of Poincaré's subjects, there was no mass movement and no quick and thorough absorption of his ideas. Two main issues seem to have stood in the way. That Poincaré had no school surely played a role. Unlike the formidable Göttingen

[1] The existence of the typescript was pointed out to me by David Chillingworth.

mathematics department that churned out young PhDs in assembly-line fashion, Poincaré's (and Hadamard's) environment had no tradition that emphasized systematic dissemination. Secondly, even though Poincaré published diligently, his writing is a pleasure to read only if one wants to be persuaded rather than be illuminated on the details of his argument. The prize paper had not been read with full care by the time the prize was assigned, and Hermite complained that it was not only impossible to get to the bottom of his reasoning by reading, but that queries would be answered only by an exasperated "it is so, it is like that".

The story of dynamics from the 1960s is quite different, and the initial explosion of activity was propelled not least by the redoubtable machines in Berkeley and Moscow that produced young practitioners and converted some reseachers from related areas. The choice of questions to investigate and publicize, combined with the computer revolution simultaneously provided for a rapid diffusion of the new ideas to the scientific community. By now dynamics as a whole has remained a (large) discipline, but has also acquired the status of a general method. Hyperbolic dynamics has been a major component in this development.

2. Hyperbolic dynamics

The primary distinction that sets apart smooth dynamics from general topological dynamics is the availability of the linearization provided by the differential; one can use the linear part of a map to draw conclusions about local behavior of the map itself. As noted in [**S-HK**] among the elliptic, parabolic, and hyperbolic situations the latter is the one where linearization is most powerful. What makes hyperbolic dynamics distinct from the other two classes is that for the linearization of a map eigenvalues off the unit circle correspond to exponential behavior under iterates, and such behavior is robust enough to produce analogous behavior for the map itself and to engender structural stability.

This local aspect of hyperbolic dynamics combined with the recurrence arising from compactness of the space provides for complex and interesting features of the global structure. In accordance with the main dichotomy between topological and measurable dynamics there are separate but related features of interest.

In contrast to the individual instability of orbits, the complicated topological dynamics of hyperbolic systems is distinguished by structural stability. Indeed, hyperbolic dynamical systems are characterized by structural stability, and to a remarkable degree a classification is possible. Furthermore, even though periodic data give a large number of moduli of differentiable conjugacy, there are interesting results about smooth conjugacy and rigidity.

On the side of measurable dynamics there is the important motivation that Hamiltonian hyperbolic flows, in particular geodesic flows of negatively curved manifolds, are ergodic (with respect to volume); this provides nontrivial classes of examples satisfying the Boltzmann ergodic hypothesis.

Altogether hyperbolic dynamical systems exhibit a remarkable combination of phenomena: Maximally sensitive dependence of an orbit on initial conditions, strong recurrence and mixing properties, many invariant measures, positive entropy, intertwining of periodic and nonperiodic orbits, an abundance of periodic points both in terms of exponential growth of their number as a function of the period and density (topologically as well as in terms of density of their δ-measures among all invariant Borel probability measures), structural stability, and the existence of a Markov model both topologically and measure-theoretically.

3. Outline of this survey

The next chapter of this survey gives definitions and basic examples as well as the theory of stable and unstable laminations. The stable/unstable manifolds theorem, also known as the Hadamard–Perron Theorem, can be taken as the base of all that follows [**KH, Fn2**]. Alternatively one can prove some central basic facts (Hartman–Grobman Theorem, expansivity, shadowing, structural stability) independently, *e.g.*, using the hyperbolic fixed point theorem (Subsection 2.1h).

Later in Chapter 2 we describe the two main methods for proving the stable/unstable manifolds theorem, the Hadamard and Perron–Irwin methods. Both rely on successive approximation, *i.e.*, the Banach Contraction Principle. The Hadamard graph transform method considers manifolds that approximate the unstable ones and applies the Banach Contraction Principle to the action of the differential on such candidates [**Hd2**]. It was specifically put forward to obviate the need for analyticity assumptions, as they had been made until then, mainly by Poincaré, and were even made later for this situation [**Bh1**, p. 45]. The Perron–Irwin method finds the stable manifold of a point by looking for bounded orbits. One considers bounded candidate constructs and an action on these by the map whose (unique) fixed point (parametrized by a point on the stable subspace) must be an orbit. Perron [**Pn1**, (13) p. 144], [**Pn2**, (10) p. 51], [**Pn4**, (30) p. 719] used a variant of Picard iteration (or variation of parameters) for solutions of differential and difference equations; Irwin [**I**] simplified this approach and combined it with the Banach Contraction Principle to obtain a short proof with strong conclusions [**Ro4**, Section 5.10], [**We, LW, Yc**].

Numerous core facts can be proved without using stable manifolds, and the survey [**Yc**], based partially on [**Sh**], shows how to take this route to present the basic theory in great brevity. It also uses the Perron–Irwin method to give a shorter proof of the stable manifold theorem. This method gives smoothness of the invariant manifolds very easily, but it provides no information on the transverse behavior of stable/unstable laminations.

Chapter 3 discusses the orbit structure of hyperbolic dynamical systems, including stability, classification and invariant measures.

Chapter 4 ends the survey of (uniformly) hyperbolic dynamics with an account of some results about smooth conjugacy and rigidity of hyperbolic dynamical systems.

The last chapter is a brief outline of the theory of nonuniformly hyperbolic systems, to be surveyed separately in [**S-BP**]. Although this theory differs from that of (uniformly) hyperbolic systems at various levels, the underlying idea is to strive for analogous results and to so do by applying the same ideas, albeit in substantially refined form. One notable distinction is that invariant measures play a much more central role and many uniform or continuous quantities associated with a hyperbolic dynamical system have measurable counterparts in the nonuniformly hyperbolic case. Our outline is a little more detailed than that in [**S-HK**], but much less so than the supplement in [**KH**]. A definitive account is forthcoming [**BKP**].

There is also a rapidly developing theory of partially hyperbolic dynamical systems, in which directions of exponential behavior are uniformly separated from a direction of subexponential behavior. The point of view in this theory is rather different, however. The central goal is to use the hyperbolic parts of the dynamics to overcome the subexponential behavior in order to arrive at global conclusions, such as ergodicity. This requires a situation that is sufficiently distinct from the product of a hyperbolic dynamical system with a nonhyperbolic one. Because of this different character and due to its present scope this area is surveyed separately [**S-B**]. We only give the briefest glimpse of it in Section 3.7.

a. Regularity. There are two regularity aspects that pervade the theory of hyperbolic dynamics and are worth previewing early. On one hand there is the regularity of the dynamics. Assuming that the diffeomorphism or flow under consideration is C^1 may suffice for the study of purely topological properties, but the typically needed minimal assumption for doing hyperbolic dynamics is that the map under consideration be $C^{1+\alpha}$ for some $\alpha > 0$, *i.e.,* have a Hölder continuous derivative [**Pu3, RoY, Bw2, PPR**]. (A map $f\colon X \to Y$ between metric spaces is said to be α-Hölder continuous if $d(f(x), f(y)) \le K d(x, y)^\alpha$ for all $x, y \in X$, Hölder, if this holds for some $\alpha > 0$.) Subsection 3.3d gives a situation where this makes a decisive difference. In the theory of nonuniform hyperbolicity this hypothesis is quite essential. It is useful to exercise some care in ensuring that such a moderate assumption suffices for any given aspect of the theory, but in this survey we set such care aside and usually assume the dynamical system to be rather smooth, *e.g.,* C^∞.

On the other hand, the structures associated with hyperbolic dynamics are Hölder continuous. More precisely, invariant structures are usually either as smooth as the map itself or Hölder continuous with an exponent usually dominated by the above α but otherwise determined from possibly subtle dynamical information. This is the case for the invariant foliations or laminations as well as for conjugacies [**HW**]. Furthermore, the class of Hölder continuous real-valued functions naturally arises as an important one. One reason is that if the corresponding terms of two sequences are exponentially close then the same goes for the image sequences under a Hölder continuous map. Another is that this class is invariant under conjugacy of hyperbolic systems because the conjugacies are Hölder continuous themselves. That Hölder continuity is the natural and prevalent regularity is related to the exponential rescaling by the dynamics in the domain and range of invariant functions or the operators used to produce the particular structure or object.

CHAPTER 2

Hyperbolic sets and stable manifolds

1. Definitions and examples

a. Hyperbolic linear maps, adapted norm. A continuous linear map $A\colon E\to E$ of a Banach space is said to be (λ,μ)-hyperbolic if $0<\lambda<\mu$ and $\mathrm{Sp}(A)\cap\{z\in\mathbb{C}\mid\lambda\le|z|\le\mu\}=\varnothing$, where $\mathrm{Sp}(A)=\{z\in\mathbb{C}\mid A_{\mathbb{C}}-z\,\mathrm{Id}\ \text{is not an automorphism}\}$ denotes the spectrum ($A_{\mathbb{C}}$ is the complexification). We say that A is hyperbolic if there exist $\lambda\in(0,1)$, $\mu>1$ such that A is (λ,μ)-hyperbolic, or equivalently, $\mathrm{Sp}(A)$ does not meet the unit circle [**DS**].

For a hyperbolic linear map there are subspaces E_s, E_u, called the stable and unstable subspaces, respectively, with $E=E_s\oplus E_u$, $A(E_s)\subset E_s$, $A(E_u)=E_u$, $\mathrm{Sp}(A_{\restriction_{E_s}})=\mathrm{Sp}(A)\cap\{|z|<1\}$, and $\mathrm{Sp}(A_{\restriction_{E_u}})=\mathrm{Sp}(A)\cap\{|z|>1\}$. An *adapted* or *Lyapunov norm* is a norm $\|\cdot\|$ equivalent to the given norm $|\cdot|$ such that $\|T_{\restriction_{E_s}}\|\le\lambda$, $\|(T_{\restriction_{E_u}})^{-1}\|\le1/\mu$ and $\|x_s+x_u\|=\max(\|x_s\|,\|x_u\|)$ for $x_s\in E_s$ and $x_u\in E_u$. (Setting $\|x_s+x_u\|^2=\|x_s\|^2+\|x_u\|^2$ works equally well.) To construct such a norm take $n\in\mathbb{N}$ sufficiently large and set

$$\|x_s\|=\sum_{i=0}^{n}|T^i(x_s)|/\lambda^i\ \text{for}\ x_s\in E_s,\qquad \|x_u\|=\sum_{i=0}^{n}|T^{-i}(x_u)|\mu^i\ \text{for}\ x_u\in E_u$$

and $\|x_s+x_u\|:=\max(\|x_s\|,\|x_u\|)$.

b. Hyperbolic sets. The definition of a hyperbolic set is traditionally given in terms of the action of iterates of the derivative extension on vectors; we give this definition after an alternative one suggested by Mather [**Mt2**], via hyperbolicity of the derivative action on vector fields, and the cone-field definition of Alekseev and Moser.

Let M be a smooth manifold, $U\subset M$ an open subset, $f\colon U\to M$ a C^1 embedding. A compact f-invariant set Λ is said to be hyperbolic if the differential of f defines a hyperbolic linear map on the space of bounded sections of $T_\Lambda M$ by $X\mapsto Df\circ X\circ f^{-1}$. The spectrum $\sigma(f)$ of the complexification of this map is called the *Mather spectrum* [**Mt2**].

An equivalent criterion can (with some diligence) be found in the work of Alekseev [**Al1**] and appears in an article of Moser [**Mos2**, Lemma 4]. It requires that for some metric there exist $\lambda<1<\mu$ and $\gamma>0$ such that for every $x\in\Lambda$ there is a decomposition $T_xM=S_x\oplus T_x$ with

$$Df_xH_x\subset\mathrm{Int}\,H_{f(x)}\ \text{and}\ {Df_x}^{-1}V_{f(x)}\subset\mathrm{Int}\,V_x,$$

where

$$H_x:=\{\xi+\eta\mid\xi\in S_x,\ \eta\in T_x,\ \|\eta\|\le\gamma\|\xi\|\},$$
$$V_x:=\{\xi+\eta\mid\xi\in S_x,\ \eta\in T_x,\ \|\xi\|\le\gamma\|\eta\|\},$$

and furthermore $\|Df_x\xi\| \geq \mu\|\xi\|$ for $\xi \in H_x$, and $\|Df_x{}^{-1}\xi\| \geq \lambda^{-1}\|\xi\|$ for $\xi \in V_{f(x)}$. H_x and V_x are called invariant horizontal and vertical *cone fields*.

Requiring existence of the (not necessarily invariant) distributions S and T is used simply as a convenient way of expressing the fact that the cones are "complementary". It suffices to exhibit the invariant cones.

This is also equivalent to existence of a Riemannian metric (called a *Lyapunov* or *adapted metric* [**Mt2**]) in an open neighborhood U of Λ such that for any $x \in \Lambda$ the sequence of differentials $(Df)_{f^n_x}\colon T_{f^n_x}M \to T_{f^{n+1}_x}M$, $n \in \mathbb{Z}$, admits a (λ, μ)-splitting, *i.e.*, there exist decompositions $T_xM = E^+_x \oplus E^-_x$ such that $DfE^\pm_x = E^\pm_{f(x)}$ and

$$\|Df\restriction_{E^-_x}\| \leq \lambda, \quad \|Df^{-1}\restriction_{E^+_x}\| \leq \mu^{-1}.$$

It immediately follows that E^+_x and E^-_x have locally constant dimension and are continuous.

Compact hyperbolic sets have interesting dynamics because the hyperbolic local picture is combined with nontrivial recurrence, which is an essentially nonlinear phenomenon.

c. Basic sets, Axiom A, Anosov diffeomorphisms. A certain completeness is often quite important:

Let Λ be a hyperbolic set for $f\colon U \to M$. If there is an open neighborhood V of Λ such that $\Lambda = \Lambda^f_V := \bigcap_{n\in\mathbb{Z}} f^n(\bar{V})$ then Λ is said to be *locally maximal* or *isolated*, and V is called an *isolating neighborhood.* One also says that Λ has *local product structure*, see Subsection 2.2h. There are several definitions of a *basic* hyperbolic set, we take it to be a topologically transitive locally maximal hyperbolic set.

For diffeomorphisms $f\colon M \to M$ we introduce *Axiom A*: $\overline{\mathrm{Per}(f)} = NW(f)$ and the nonwandering set $NW(f)$ is hyperbolic.

The horseshoe (below) is locally maximal by construction. So is the following class of examples, which was one of the primary motivations to study hyperbolic dynamics:

A C^1 diffeomorphism $f\colon M \to M$ of a compact manifold M is said to be an *Anosov diffeomorphism* [**A1**] if M is a hyperbolic set for f.

Clearly, any sufficiently small C^1-perturbation of an Anosov diffeomorphism is an Anosov diffeomorphism. (For other hyperbolic sets the existence of a nontrivial invariant set for perturbations is not as evident, see Subsection 3.5a.) The Anosov situation is special in a substantial way: Except in the Anosov case hyperbolic sets usually have zero Lebesgue measure (Subsection 3.3d).

d. Examples.

1. *The Smale horseshoe.* The prototypical example of a hyperbolic set is Smale's original "horseshoe" [**S2**] described in [**S-HK**] and [**KH**]. Let Δ be a rectangle in $\mathbb{R}^2$ and $f\colon \Delta \to \mathbb{R}^2$ a diffeomorphism of Δ onto its image such that the intersection $\Delta \cap f(\Delta)$ consists of two "horizontal" rectangles Δ_0 and Δ_1 and the restriction of f to the components $\Delta^i \subset f^{-1}(\Delta)$, $i = 0, 1$, of $f^{-1}(\Delta)$ is a hyperbolic affine map, contracting in the vertical direction and expanding in the horizontal direction. The maximal invariant subset of Δ is $\Lambda = \bigcap_{n=-\infty}^{\infty} f^{-n}(\Delta)$. This is the product of two Cantor sets, hence a Cantor set itself. One can take the Euclidean metric, $\lambda = 1/2$, $\mu = 2$, and the splitting into the horizontal and the vertical directions as the hyperbolic splitting.

An important manifestation of (nonlinear) horseshoes in modern times contributed to the development of hyperbolic dynamics. When studying unpredictability in relaxation oscillations of certain radar circuits (tuned to outside normal operating parameters), Cartwright and Littlewood [**CL, C, Lw**] came up with a careful and somewhat elaborate mathematical description of the situation. When Smale conjectured that structural stability necessitated finiteness of the set of periodic points [**S1**], Levinson pointed out that systems such as that of Cartwright and Littlewood are structurally stable but have infinitely many periodic points. Upon careful study of Levinson's work [**Lv**] Smale saw that the essential geometric ingredient is a picture that he eventually distilled into the horseshoe [**S3, S2**].

2. *Transverse homoclinic points and horseshoes.* The appearance of horseshoes in mathematical models of real-world phenomena is quite widespread. Indeed, in a sense this is *the* mechanism for the production of chaotic behavior (Theorem 5.8.1), at least in dimension two. In disguise, one of the earliest appearances of this phenomenon occurred in the prize memoir of Poincaré [**Pc**], where homoclinic tangles gave a first glimpse at the serious dynamical complexity that can arise in the three-body problem in celestial mechanics. Homoclinic tangles always produce horseshoes by the Smale–Birkhoff Theorem [**S3**], so in trying to solve the three-body problem Poincaré essentially discovered the possibility of nontrivial hyperbolic behavior. A related appearance of horseshoes in this context is in the work of Alekseev, who used their presence to show that capture of celestial bodies can indeed occur [**AKK, Al2**].

We give a brief description of how transverse homoclinic points give rise to horsehoes. A full treatment can be found in [**KH**]. Consider the hyperbolic linear map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (2x, y/2)$ in a neighborhood of the origin. The segment of the y-axis consists of points asymptotic to the origin in positive time and the segment of the x-axis consists of points asymptotic to the origin in negative time, while all other points move along hyperbolas $xy = \mathrm{const}$. If we extend our map such that the preimage of the y-axis and the image of the x-axis intersect transversely at a point p then p is called a *transverse homoclinic point* for the fixed point 0. In this case $f^n(p) \to 0$ as $|n| \to \infty$. Since these unstable and stable manifolds of 0 (see Section 2.2) are invariant under f, the images $f^n(q)$ are also homoclinic points, *i.e.,* intersection points of the unstable and stable manifolds of the origin. Since the intersection at q is transverse and f is a diffeomorphism, the same is true at $f^n(q)$ for any n. We thus immediately obtain a countable number of transverse homoclinic points. Between any two of these we have "homoclinic loops". f maps these loops to each other, *e.g.,* the loop between q and r to that between $f(q)$ and $f(r)$. Since the unstable manifold has no self-intersections, we get increasingly thin loops accumulating on the unstable manifold, and likewise for the stable manifold. Thus, these invariant curves (the stable and unstable manifolds of 0), produce a complex web of tangles [**KH**, Figure 6.5.2] that necessitates the presence of horseshoes near 0.

Poincaré encountered this situation in his revised prize memoir [**Pc, B-G**], and it illustrated the potential orbit complexity in the three-body problem. Birkhoff subsequently proved that there are infinitely many periodic points in a neighborhood of the origin [**Bh2**, Section 9], but he did not find all the periodic orbits produced by a horseshoe. While Cartwright and Littlewood were aware of Poincaré's work, Smale was not.

3. *The Smale attractor.* To obtain the *Smale attractor* [**S-HK, KH, S4**] or *solenoid* on the solid torus $M = S^1 \times D^2$, where D^2 is the unit disk in $\mathbb{R}^2$, choose coordinates (φ, x, y) such that $\varphi \in S^1$ and $x^2 + y^2 \le 1$ and define $f\colon M \to M$ by

$$f(\varphi, x, y) = \Big(2\varphi, \frac{1}{10}x + \frac{1}{2}\cos\varphi, \frac{1}{10}y + \frac{1}{2}\sin\varphi\Big).$$

Then $\Lambda := \bigcap_{l \in \mathbb{N}_0} f^l(M)$ is an attractor on which f is expanding. Locally it is the product of a Cantor set with an interval, but it is connected. The stable manifolds are the sections $C = \{\theta\} \times D^2$, the unstable manifold of each point is entirely contained in the attractor.

Smale called this map the DE-map, for “derived from expanding”. The attractor is the natural extension [**S-HK**] of the double self-covering $x \mapsto 2x$ of the circle.

4. *Toral automorphisms.* Any automorphism F_L induced on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ by a hyperbolic linear map L of R^n with integer entries and determinant ± 1 is an Anosov diffeomorphism. There is a Euclidean norm in $\mathbb{R}^n$ that makes L contracting in $E^-(L)$ and expanding in $E^+(L)$; it projects to $\mathbb{T}^n$, where we have an invariant splitting into subspaces parallel to $E^+(L)$ and $E^-(L)$. Simple instances are $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and its square $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, depicted here. These examples are amenable to explicit computation. For instance, the entropy of the latter automorphism is $\log \dfrac{3+\sqrt{5}}{2}$, the logarithm of the maximal eigenvalue.

5. *Automorphisms of infranilmanifolds.* The only known manifolds that support Anosov diffeomorphisms are infranilmanifolds (of which tori are a special case). On these one can construct hyperbolic automorphisms [**KH, S4**]. Suppose G is a simply connected Lie group and Γ is a discrete cocompact subgroup. If $F\colon G \to G$ is an automorphism such that $F(\Gamma) = \Gamma$ (hence F projects to $\Gamma\backslash G$) and $DF_{|_{\mathrm{Id}}}$ is hyperbolic then there is a splitting of the Lie algebra $\mathcal{L}(G) = T_{\mathrm{Id}}G = E^+ \oplus E^-$ and a norm on $\mathcal{L}(G)$ such that $DF^{-1}\restriction_{E^+}$ and $DF\restriction_{E^-}$ are contractions. (This implies that G is nilpotent.) Via translations we obtain a splitting, which is a hyperbolic splitting for F. By construction this splitting and this norm are invariant under left translations, so they induce a splitting and a norm on the compact quotient $\Gamma\backslash G$. The factor $f\colon \Gamma\backslash G \to \Gamma\backslash G$ of F is then an Anosov diffeomorphism. Specific examples of such infranilmanifoldautomorphisms are given in [**KH, S4**].

6. *Further examples.* Other classical examples are the DA-map (“derived from Anosov”) and the Plykin attractor [**KH**]. As we note below, perturbations of all of the above examples are examples as well. Therefore, we have C^1-open sets of hyperbolic systems.

7. *Repellers.* E_x^- is determined by the positive semiorbit of x and can thus naturally be defined in the noninvertible case, but E_x^+ is determined by the negative semiorbit of x, which is not uniquely defined for noninvertible maps, making the notion of hyperbolicity less straightforward. However, there is no ambiguity in choosing the expanding part if it is the entire tangent space:

Let $f\colon U \to M$ be C^1. A compact invariant set Λ is said to be a *hyperbolic repeller* if there exists a Riemannian metric in a neighborhood of Λ such that $\|Df(v)\| > \|v\|$ for all $v \in T_\Lambda M$.

For an expanding map (*i.e.*, a map with $\|Df(v)\| \geq \lambda\|v\|$ for some $\lambda > 1$ and all $v \in TM$) the whole manifold is a hyperbolic repeller. The map $x \mapsto 2x \pmod 1$ on S^1 is a standard example. The maximal invariant sets in $[0,1]$ for quadratic maps $ax(1-x)$ with $a > 4$ are examples of Cantor sets that are hyperbolic repellers.

e. Hyperbolic sets for flows. Let M be a smooth manifold, $\varphi\colon \mathbb{R} \times M \to M$ a smooth flow, and $\Lambda \subset M$ a compact φ^t-invariant set. The set Λ is said to be a *hyperbolic set for the flow* φ^t if there exist a Riemannian metric on an open neighborhood U of Λ and $\lambda < 1 < \mu$ such that for all $x \in \Lambda$ there is a decomposition $T_xM = E_x^0 \oplus E_x^+ \oplus E_x^-$ with $\dfrac{d}{dt}_{|t=0}\varphi^t(x) \in E_x^0 \smallsetminus \{0\}$, $\dim E_x^0 = 1$, $D\varphi^t E_x^\pm = E_x^\pm$, and

$$\|D\varphi^t\restriction_{E_x^-}\| \leq \lambda^t, \quad \|D\varphi^{-t}\restriction_{E_x^+}\| \leq \mu^{-t}.$$

Equivalently, a compact φ^t-invariant set $\Lambda \subset M$ is hyperbolic if for some metric there exist constants $\lambda < 1 < \mu$ such that for all $x \in \Lambda$ there is a decomposition $T_xM = E_x^0 \oplus S_x \oplus T_x$ (in general not $D\varphi^t$-invariant), a family of horizontal cones $H_x \supset S_x$

associated with the decomposition $S_x \oplus (E^0_x \oplus T_x)$, and a family of vertical cones $V_x \supset T_x$ associated with the decomposition $(S_x \oplus E^0_x) \oplus T_x$ such that for $t > 0$

$$D\varphi^t H_x \subset \operatorname{Int} H_{\varphi^t(x)}, \qquad D\varphi^{-t} V_x \subset \operatorname{Int} V_{\varphi^{-t}(x)},$$
$$\frac{d}{dt}\|D\varphi^t \xi\| \geq \|\xi\| \log \mu \quad \text{for} \quad \xi \in H_x,$$
$$\frac{d}{dt}\|D\varphi^{-t} \xi\| \geq \|\xi\| \log \lambda \quad \text{for} \quad \xi \in V_x \ .$$

A C^1 flow $\varphi^t \colon M \to M$ on a compact manifold M is said to be an *Anosov flow* [**A1**] if M is a hyperbolic set for φ^t.

Hyperbolicity does not depend on the parametrization of time: If Λ is a hyperbolic set for φ^t and ψ^t is a time change of φ^t then Λ is hyperbolic for ψ^t.

f. Examples.

1. *Geodesic flows.* The central example of Anosov flows is provided by geodesic flows—these were an important motivation for developing the hyperbolic theory [**A1**]. Given a Riemannian manifold M one can define the geodesic flow g^t on the unit tangent bundle $T^1M := \{v \in TM \mid \|v\| = 1\}$ by $\gamma^t(v) = \dot\gamma_v(t)$, where γ_v is the geodesic defined by $\dot\gamma_v(0) = v$. In other words, one follows the geodesic in the direction v for time t and then takes the tangent vector there as the image. Equivalently, one can describe the flow as the Hamiltonian flow for free particle motion (no potential) on M restricted to the energy level 1/2. (Considering a different energy level or, equivalently, vectors of another fixed length changes only the speed along the geodesics, hence amounts to a constant time change [**S-HK**] only.)

If the sectional curvature is negative everywhere and the manifold is compact then the geodesic flow is an Anosov flow. To see this one can verify the definition in terms of contracting and expanding subspaces [**Kl2**], which leads to a study of Jacobi fields, the Jacobi equation, and the associated Riccati equation. This has the advantage that, done carefully, it establishes connections between pinching control of curvature and "bunching" control of contraction and expansion rates of the flow (this plays a role in the transverse regularity of the invariant laminations, which are called the horospheric foliations in this context; see Section 2.3). On the other hand, one can verify the cone criterion in the definition of hyperbolicity, which is more geometric [**KH**]. We briefly outline this approach. The basic connection between Jacobi fields and dynamics can be found in [**S-K**]. Jacobi fields along a geodesic are vector fields generated by variations of the geodesic. Therefore their growth measures the degree of divergence between nearby geodesics. Thus Jacobi fields represent the action of the differential of the geodesic flow (the geodesic spray) on one hand, and are given in terms of curvature information on the other hand. Denote by R the curvature tensor. Jacobi fields $Y\colon t \mapsto Y(t) \in T_{\gamma(t)}M$ along a geodesic $\gamma\colon \mathbb{R} \to M$ are obtained as solutions of the Jacobi equation

$$\ddot Y(t) + K(t)Y(t) = 0,$$

where dots denote differentiation with respect to t and $K(t) := R(\dot\gamma(t), \cdot)\dot\gamma(t)$. They arise from variations of geodesics, which causes their behavior to reflect the dynamics of the geodesic flow. For $p \in M$, $v \in T_pM$ denote by γ_v the geodesic with $\gamma_v(0) = p$, $\dot\gamma_v(0) = v$. Then there are isomorphisms $\psi_v\colon T_vTM \to T_pM \oplus T_pM$, $\xi \mapsto (x, x')$ such that $\psi_{g^tv}(Dg^t\xi) = (Y(t), \dot Y(t))$, where Y is the Jacobi field along γ_v with $Y(0) = x$ and $\dot Y(0) = x'$. This allows us to describe the dynamics of the geodesic flow in terms of the evolution of Jacobi fields and to speak of the action g^t (or Dg^t, rather) on Jacobi fields. In order to establish that the geodesic flow in SM is an Anosov flow it is sufficient to obtain invariant cones. To that end we study the Jacobi equation $\ddot{Y} + KY = 0$ with a negative-semidefinite symmetric operator K. Introduce a norm on $T_pM \oplus T_pM$ by $\|u, v\| := \sqrt{\langle u, u\rangle + \langle v, v\rangle}$ for $u, v \in T_pM$. Then the family of cones C given by $\dfrac{\langle Y, \dot{Y}\rangle}{\|Y, \dot{Y}\|^2} \geq 0$ is invariant because if $\dfrac{\langle Y, \dot{Y}\rangle}{\|Y, \dot{Y}\|^2} = 0$ then

$$\frac{d}{dt}\frac{\langle Y, \dot{Y}\rangle}{\|Y, \dot{Y}\|^2} = \frac{(\langle \dot{Y}, \dot{Y}\rangle + \langle \ddot{Y}, Y\rangle)\|Y, \dot{Y}\|^2 - 2\langle Y, \dot{Y}\rangle(\langle Y, \dot{Y}\rangle + \langle \ddot{Y}, \dot{Y}\rangle)}{\|Y, \dot{Y}\|^4}$$
$$= \frac{\langle \dot{Y}, \dot{Y}\rangle - \langle KY, Y\rangle}{\|Y, \dot{Y}\|^2} \geq \frac{\langle \dot{Y}, \dot{Y}\rangle}{\|Y, \dot{Y}\|^2} \geq 0.$$

This does not quite prove hyperbolicity, but then we only used that the curvature is nonpositive. With negative curvature and slightly more narrow cones one obtains strict invariance and exponential expansion in the cones [**KH**].

Hyperbolicity in geodesic flows is the subject of the survey [**S-K**]. To the extent that there is any overlap, that survey is usually the better place to consult. For this and other aspects of geodesic flows see also [**Pt3**].

2. *Suspensions.* Given a diffeomorphism $f\colon M \to M$ of a manifold one can define the *suspension flow* [**S-HK**] on $M \times [0, 1]/(x, 1) \sim (f(x), 0)$ as the flow integrating the vertical vector field. Its return map to $M \times \{0\}$ gives the original diffeomorphism. If f has a compact invariant hyperbolic set Λ then $\Lambda \times [0, 1]/(x, 1) \sim (f(x), 0)$ is a compact invariant hyperbolic set for the suspension flow. Many recurrence properties of the map f are clearly inherited by the suspension flow, but this is not true for topological mixing; suspension flows are never mixing (consider $U = V = M \times (0, \epsilon)$ or note that the constant speed flow on the circle is an obvious topological factor and is not mixing). Transitive Anosov flows are either mixing, such as geodesic flows, or a suspension (Subsection 3.3b).

Given a smooth function $\varphi\colon M \to \mathbb{R}^+$ one may also consider the *special flow over f under φ* [**S-HK**] defined on

$$M_\varphi := \{(x, y) \in M \times \mathbb{R} \mid 0 \leq y \leq \varphi(x)\}/(x, \varphi(x)) \sim (f(x), 0)$$

by integrating the vertical vector field $\dfrac{\partial}{\partial y}$. The function $\varphi \equiv 1$ gives the suspension; whether mixing is inherited by a special flow depends on f and φ. If f is a mixing Anosov diffeomorphism then the special flow under φ is mixing if and only if φ is not cohomologous to a constant (Anosov alternative).

3. *Further examples.* An obvious way of producing new flows from old ones is to impose a time change. In most cases this will change a geodesic flow to a nongeodesic one, for example. Similarly to diffeomorphism situation one can also consider perturbations, but by structural stability this does not change the orbit structure. Nevertheless, some particular kind of perturbations are of interest in regard to finer information. Adding a "magnetic force term" to a geodesic flow is an example.

Unlike the discrete-time case, however, there is at present no hope, much less any strategy, for classifying Anosov flows. The reason is that there are several distinctive examples available. An important one is that by Franks and Williams, an Anosov flow in dimension three that is not transitive [**FrW**], in contrast to all known discrete-time examples. They call it anomalous. A much more recent new class of examples is due to Foulon [**Fo2**]. These are smooth contact Anosov flows on closed three-manifolds, as are geodesic flows on compact negatively curved surfaces. But in these examples the three-manifolds are *not* the unit tangent bundle of any surface.

g. The Banach Contraction Principle. The truly basic technical fact underlying much of the development of hyperbolic dynamics, including the existence of stable and unstable manifolds, is the Banach Contraction Principle, itself a description of a class of simple dynamical systems. It is worth recalling in a form that includes basic facts about the dependence of the fixed point on the contraction [**Yc**].

Let X, Y be metric spaces, X complete, $\lambda \in [0,1)$ and $T\colon Y \times X \to X$ such that $d(T_y(x), T_y(x')) \le \lambda d(x,x')$. Then each T_y has a unique fixed point $\varphi(y)$. Furthermore, $d(\varphi(y), x) \le d(x, T_y(x))/(1-\lambda)$ and hence

$$d(\varphi(y), \varphi(y')) \le \frac{d(T_{y'}(\varphi(y')), T_y(\varphi(y')))}{1-\lambda} \text{ for } y, y' \in Y.$$

Thus φ is continuous if T is and α-Hölder if T is. If $\mathrm{Lip}(T) = l < 1$ then

$$d(\varphi(y), \varphi(y')) = d(T_y(\varphi(y)), T_{y'}(\varphi(y'))) \le l \max(d(y,y'), d(\varphi(y), \varphi(y'))) = l d(y,y'),$$

so $\mathrm{Lip}(\varphi) = l$. If T is C^r then so is φ and $D_y\varphi = (\mathrm{Id} - D^X_{(y,\varphi(y))}T)^{-1} \circ D^Y_{(y,\varphi(y))}T$, where D^X and D^Y are partial derivatives.

This is proved by bounding the distance between successive iterates of a point by a geometric series; the last conclusion uses the Implicit Function Theorem.

The Banach Contraction Principle is usually applied by constructing from the underlying hyperbolic dynamics an action on a space of auxiliary "candidate" objects, which is a contraction. The fixed point then gives the desired item.

h. The hyperbolic fixed point theorem. Let $0 < \lambda < 1 < \mu$, E a Banach space, A a (λ, μ)-hyperbolic linear continuous map, $\|\cdot\|$ an adapted norm, $f\colon E \to E$ a Lipschitz continuous map with $\epsilon := \mathrm{Lip}(f - A) < \epsilon_0 := \min(1-\lambda, 1-\mu^{-1})$. Then f has a unique fixed point $p \in E$ and $\|p\| < \|f(0)\|/(\epsilon_0 - \epsilon)$.

Some proofs that rely on the Banach Contraction Principle can be made more direct and shortened a little (by simplifying the construction of the action on the auxiliary space) when one employs this fixed point result (itself a consequence of the Banach Contraction Principle). The Hartman–Grobman Theorem, expansivity, the Anosov Closing Lemma, shadowing, and structural stability can be obtained this way [**S-HK**]. For the moment it is also interesting as a first illustration of the pervasive C^1-stability observed in hyperbolic dynamics because many objects associated with a hyperbolic dynamical system (invariant structures, conjugacies) are often obtained via a fixed point statement and, like the fixed point p here, depend nicely on the dynamical system. This is used in [**Yc**].

2. Stable manifolds

Stable and unstable manifolds are the most prominent feature of hyperbolic dynamical systems. They are closely connected with the success of linearization and localization, both because they are produced using linearization and localization, and because they are themselves tools in local, semilocal and global analysis. While one can base most of the structural analysis of hyperbolic dynamical systems on these invariant laminations, it is also possible to derive a substantial collection of core facts without these [**S-HK, Yc**]. Especially in the theory of nonuniformly hyperbolic systems it has been useful as well to substitute "admissible" manifolds for stable ones. These are easier to obtain and turn out to be sufficient for many purposes.

a. The Stable Manifold Theorem. Let $0 < \lambda < \eta < \mu$, E a Banach space, A a (λ, μ)-hyperbolic linear continuous map, $\|\cdot\|$ an adapted norm, $f\colon E \to E$ a Lipschitz continuous map with $\epsilon := \mathrm{Lip}(f - A) < \epsilon_0 := \min(\eta - \lambda, \mu - \eta)$. Then the *$\eta$-contracting manifold*

$$W^s_\eta(f) := \{x \in E \mid \sup_{n\in\mathbb{N}} \eta^{-n}\|f^n(x)\| < +\infty\}$$

is the graph of a contraction $\varphi\colon E_s \to E_u$ with $\varphi(0) = 0$, $\mathrm{Lip}(f\restriction_{W^s_\eta(f)}) \le \epsilon + \lambda$ and $\lim_{n\to\infty} \eta^{-n}\|f^n(x)\| = 0$ for all $x \in W^s_\eta(f)$. If $\eta < 1$ and f is C^r with $1 \le r \le \infty$ then $\varphi \in C^r$; if $D_0 f = A$ then $D_0\varphi = 0$, *i.e.*, W^s_η is tangent to E_s. Note that if $\eta \ge 1$ then the action of f on $W^s_\eta(f)$ may not be contracting.

Complementary η-expanding manifolds W^u_η are defined and obtained as η^{-1}-contracting manifolds of f^{-1}. Note that this result does not make assumptions on the size of λ or μ relative to 1; thus we obtain stable leaves for diffeomorphisms and strong stable leaves for flows ($\eta = 1$), weak stable leaves for flows ($\lambda = 1$) and fast stable leaves when there is a further spectral gap inside the unit circle. (This result also has a conclusion for $\lambda > 1$, but this conclusion is lost when one applies the localization procedure that gives results on compact manifolds, such as stable and unstable laminations [**KH, Yc**]).

We apply this result to hyperbolic sets; already in the present form one can appreciate that it guarantees existence of nonlinear objects corresponding to linear objects with exponential behavior. We sketch two methods of proof.

b. The Perron–Irwin method. (See also [**Pn1, I, We, LW, FHY, Yc**], [**Ro4**, Section 5.10].) To find W^s we write $f = (f_s, f_u)$,

$$\mathcal{E}_s := \{(x^n_s)_{n\in\mathbb{N}} \mid x^n_s \in E_s,\ \|(x^n_s)_{n\in\mathbb{N}}\| := \sup_n \|x^n_s\| < \infty\},$$
$$\mathcal{E}_u := \{(x^n_u)_{n\in\mathbb{N}_0} \mid x^n_u \in E_u,\ \|(x^n_u)_{n\in\mathbb{N}}\| := \sup_n \|x^n_u\| < \infty\},$$

and define $\theta\colon E_s \times \mathcal{E}_s \times \mathcal{E}_u \to \mathcal{E}_s \times \mathcal{E}_u$, $(x_s, (x^n_s)_{n\in\mathbb{N}}, (x^n_u)_{n\in\mathbb{N}_0}) \mapsto ((y^n_s)_{n\in\mathbb{N}}, (y^n_u)_{n\in\mathbb{N}_0})$, where $x^0_s := x_s$, $y^{n+1}_s = f_s(x^n_s, x^n_u)$ and $y^n_u = x^n_u + (D(f_u\restriction_{E_u}))^{-1}(x^{n+1}_u - f_u(x^n_s, x^n_u))$ for $n \in \mathbb{N}_0$. The sequence spaces can be thought of as candidates for orbits; indeed, a fixed point of θ is a bounded orbit (parametrized by its initial stable coordinate x_s). To get the theorem one therefore shows that θ is a contraction depending smoothly on x_s; this gives existence, uniqueness, and smoothness of stable manifolds.

c. The Hadamard graph transform method. (See also [**HPS, S-HK, KH, Hd2**].) To find W^u (which is W^s for f^{-1}) choose $\gamma \in (0, 1)$ (depending on contraction and expansion rates), and consider γ-Lipschitz maps $\varphi\colon E_u \to E_s$ with $\varphi(0) = 0$ (candidates for the unstable manifold). The *graph-transform* or *Riccati operator* $\mathcal{F}$ defined by

$$\mathrm{graph}(\mathcal{F}(\varphi)) = f(\mathrm{graph}(\varphi))$$

is a contraction with respect to the metric

$$d(\varphi, \varphi') := \sup_{x\in E_u\smallsetminus\{0\}} \|\varphi(x) - \varphi'(x)\|/\|x\|$$

and W^u is the graph of the unique fixed point.

This method revolves around the invariance of cone fields, which was introduced in the definition of a hyperbolic set.

d. The Hadamard–Perron Theorem. The result that can be taken as the base of stable manifold theory for both the uniformly and nonuniformly situations is the Hadamard–Perron Theorem, which we describe here in the same form as in [**S-HK, KH**]. It can be proved by the method of Hadamard [**KH**, Section 6.2d] or by that of Perron–Irwin.

Let $\lambda < \mu$ and choose $0 < \gamma < \min\big(1, \sqrt{\mu/\lambda} - 1\big)$ and

$$0 < \delta < \min\left(\frac{\mu - \lambda}{\gamma + 2 + 1/\gamma}, \ \frac{\mu - (1+\gamma)^2\lambda}{(1+\gamma)(\gamma^2 + 2\gamma + 2)}\right).$$

For $r \geq 1$ and for each $m \in \mathbb{Z}$ let $f_m \colon \mathbb{R}^n \to \mathbb{R}^n$ be a (surjective) C^r diffeomorphism such that $f_m(x,y) = (A_m x + \alpha_m(x,y),\ B_m y + \beta_m(x,y))$ for $(x,y) \in \mathbb{R}^k \oplus \mathbb{R}^{n-k}$, where $A_m \colon \mathbb{R}^k \to \mathbb{R}^k$ and $B_m \colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ are linear maps with $\|A_m^{-1}\| \leq \mu^{-1}$, $\|B_m\| \leq \lambda$ and $\alpha_m(0) = 0$, $\beta_m(0) = 0$, $\|\alpha_m\|_{C^1} < \delta$, $\|\beta_m\|_{C^1} < \delta$. Then there are

1. a unique family $\{W_m^+\}_{m\in\mathbb{Z}}$ of k-dimensional C^1 manifolds

   $$W_m^+ = \{(x, \varphi_m^+(x)) \mid x \in \mathbb{R}^k\} = \operatorname{graph}\varphi_m^+$$

   and
2. a unique family $\{W_m^-\}_{m\in\mathbb{Z}}$ of $(n-k)$-dimensional C^1 manifolds

   $$W_m^- = \{(\varphi_m^-(y), y) \mid y \in \mathbb{R}^{n-k}\} = \operatorname{graph}\varphi_m^-,$$

where $\varphi_m^+ \colon \mathbb{R}^k \to \mathbb{R}^{n-k}$, $\varphi_m^- \colon \mathbb{R}^{n-k} \to \mathbb{R}^k$, $\sup_{m\in\mathbb{Z}} \|D\varphi_m^\pm\| < \gamma$, and the following properties hold:

1. $f_m(W_m^-) = W_{m+1}^-$, $\quad f_m(W_m^+) = W_{m+1}^+$.
2. $\|f_m(z)\| < \lambda'\|z\|$ for $z \in W_m^-$, $\|f_{m-1}^{-1}(z)\| < (\mu')^{-1}\|z\|$ for $z \in W_m^+$, where $\lambda' := (1+\gamma)\,(\lambda + \delta(1+\gamma)) < \dfrac{\mu}{1+\gamma} - \delta =: \mu'$.
3. Let $\lambda' < \nu < \mu'$. If $\|f_{m+L-1} \circ \cdots \circ f_m(z)\| < C\nu^L\|z\|$ for all $L \geq 0$ and some $C > 0$ then $z \in W_m^-$.
   Similarly, if $\|f_{m-L}^{-1} \circ \cdots \circ f_{m-1}^{-1}(z)\| \leq C\nu^{-L}\|z\|$ for all $L \geq 0$ and some $C > 0$ then $z \in W_m^+$.

Finally, in the hyperbolic case $\lambda < 1 < \mu$, the families $\{W_m^+\}_{m\in\mathbb{Z}}$ and $\{W_m^-\}_{m\in\mathbb{Z}}$ consist of C^r manifolds.

This result can be applied to hyperbolic sets by looking at one orbit at a time and taking the sequence of maps to be coordinate representations of the diffeomorphism at a given iterate. The uniformity of the estimates for different orbits gives uniform results. But this uniformity is not an essential ingredient and accordingly, the Hadamard–Perron Theorem is a central device for the nonuniformly hyperbolic situation as well.

The application to flows goes via time-one maps and gives both strong and weak (un)stable manifolds.

e. Stable and unstable laminations. Let Λ be a hyperbolic set for a C^1 diffeomorphism $f \colon V \to M$ such that $Df\restriction_\Lambda$ admits a (λ, μ)-splitting with $\lambda < 1 < \mu$ (see the definition of hyperbolic set). Then for each $x \in \Lambda$ there is a pair of embedded C^1 discs $\widetilde{W}^s(x)$, $\widetilde{W}^u(x)$, called the *local stable manifold* and the *local unstable manifold* of x, respectively, such that

1. $T_x\widetilde{W}^s(x) = E_x^-$, $\quad T_x\widetilde{W}^u(x) = E_x^+$;
2. $f(\widetilde{W}^s(x)) \subset \widetilde{W}^s(f(x))$, $f^{-1}(\widetilde{W}^u(x)) \subset \widetilde{W}^u(f^{-1}(x))$;

3. for every $\delta > 0$ there exists $C(\delta)$ such that for $n \in \mathbb{N}$

$$d(f^n(x), f^n(y)) < C(\delta)(\lambda+\delta)^n d(x,y) \text{ for } y \in \widetilde{W}^s(x),$$
$$d(f^{-n}(x), f^{-n}(y)) < C(\delta)(\mu-\delta)^{-n} d(x,y) \text{ for } y \in \widetilde{W}^u(x);$$

4. there exists $\beta > 0$ and a family of neighborhoods O_x containing the ball around $x \in \Lambda$ of radius β such that

$$\widetilde{W}^s(x) = \{y \mid f^n(y) \in O_{f^n(x)}, \qquad n = 0, 1, 2, \dots\},$$
$$\widetilde{W}^u(x) = \{y \mid f^{-n}(y) \in O_{f^{-n}(x)}, \quad n = 0, 1, 2, \dots\}.$$

5. Global stable and unstable manifolds

$$W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(\widetilde{W}^s(f^n(x))), \quad W^u(x) = \bigcup_{n=0}^{\infty} f^n(\widetilde{W}^u(f^{-n}(x)))$$

are well-defined and can be characterized topologically:

$$W^s(x) = \{y \in U \mid d(f^n(x), f^n(y)) \xrightarrow[n\to\infty]{} 0\},$$
$$W^u(x) = \{y \in U \mid d(f^{-n}(x), f^{-n}(y)) \xrightarrow[n\to\infty]{} 0\}.$$

6. If $W^s(x) \cap W^s(y) \neq \varnothing$, then $W^s(x) = W^s(y)$.
7. Denote by $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ the ϵ-balls in $W^s(x)$ and $W^u(x)$ (*local stable and unstable manifolds*). Then there exists an $\epsilon > 0$ such that for any $x, y \in \Lambda$ the intersection $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of at most one point $[x, y]$ and there exists a $\delta > 0$ such that whenever $d(x, y) < \delta$ for some $x, y \in \Lambda$ then $W^s_\epsilon(x) \cap W^u_\epsilon(y) \neq \varnothing$.
8. Let Λ be a compact hyperbolic set for $f\colon U \to M$. Then there exists an open neighborhood V_0 of Λ and $\alpha_0 > 0$ such that whenever $x, y \in \Lambda$ and $\{z\} = W^s(x) \cap W^u(y) \subset V_0$ then for any $\xi \in T_z W^s(x)$ and $\eta \in T_z W^u(y)$ the angle between ξ and η is greater than α_0.

Global stable and unstable manifolds are usually immersed in the phase space in a complicated way. Those for 0 under the action of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on $\mathbb{T}^2$ are projections of eigenlines, hence dense (see also Subsubsection 2.1d2).

This is the stable manifold theorem for hyperbolic sets: there are stable and unstable manifolds through every point of the hyperbolic set, and these form a pair of Hölder continuous (see Section 2.3) transverse laminations with smooth leaves. One can obtain it from the Hadamard–Perron Theorem, or by applying the stable manifold theorem to the action on vector fields via a localization procedure [**KH**, Lemma 6.2.7], [**Yc**, Section 2.4].

f. Fast leaves. The statement of the main stable manifold theorem and that of the Hadamard–Perron Theorem carefully avoided specifying the sizes of the rates λ and μ relative to 1. The benefit is that if $\mu < 1$ and the linear map A is (λ, μ)-hyperbolic as well as hyperbolic then for f we obtain a *fast stable manifold*

$$W^s_\eta(f) := \{x \in E \mid \sup_{n \in \mathbb{N}} \eta^{-n} \|f^n(x)\| < +\infty\} \subset W^s_1(f) = W^s(f).$$

Therefore, if the part of the spectrum of (the complexification of) A inside the unit circle consists of several separated annuli then for sufficiently nearby f we obtain a nested collection of stable manifolds. More specifically, if the spectrum of A is contained in

$\bigcup_{i=-k}^{l}\{z \in \mathbb{C} \mid \mu_i \le |z| \le \lambda_{i+1}\}$ with $\lambda_i < \eta_i < \mu_i$ and $\eta_0 = 1$ then we obtain fast stable manifolds

$$W^s_{\eta_i}(f) := \{x \in E \mid \sup_{n\in\mathbb{N}} \eta_i^{-n}\|f^n(x)\| < +\infty\} \subset W^s_{\eta_{i+1}}(f)$$

for $-k \le i < 0$, which define the stable filtration, and likewise an unstable filtration.

If the action of a diffeomorphism on vector fields (on a hyperbolic set, complexified) satisfies the same spectral assumption then there are subbundles E^i associated with the spectral annuli and characterized by ever faster contraction/expansion rates and one obtains a stable and unstable filtration of the invariant laminations, where $TW^s_{\eta_i} = \bigoplus_{j\le i} E^j$.

For the case of Anosov diffeomorphisms Mather showed that if nonperiodic points are dense then the Mather spectrum (Subsection 2.1b) is of the form

$$\sigma(f) = \bigcup_{i=-k}^{l}\{z \in \mathbb{C} \mid \mu_i \le |z| \le \lambda_{i+1}\}$$

with $\lambda_i < \mu_i$. According to a result by Pesin [**Pe1**], for all $\epsilon > 0$ there is a C^1 neighborhood U of f such that

$$\sigma(g) \subset \bigcup_{i=-k}^{l}\{z \in \mathbb{C} \mid \mu_i - \epsilon \le |z| \le \lambda_{i+1} + \epsilon\}$$

for all $g \in U$, *i.e.*, the Mather spectrum is semicontinuous.

g. Slow leaves. From the preceding discussion the question arises whether the subbundles E^i associated to the annuli of the Mather spectrum are tangent to invariant leaves. Except for the smallest and largest indices i the Stable Manifold Theorem gives no pertinent information. In general, there are assumptions on the sizes of the spectral gaps that give invariant local C^r manifolds (r depending on the spectrum) tangent to E^i, but these cannot always be glued together to give a foliation, *i.e.*, each point has such a slow manifold, but these are not equivalence classes [**JLP**]. For perturbations of toral automorphisms satisfying the proper spectral assumptions the slow subbundles integrate to continuous slow foliations with C^r leaves [**LW**], where r is again related to spectral parameters and cannot be improved. In contrast to the fast leaves, the slow leaves are much less regular than the diffeomorphism. Their degree of differentiability depends on spectral conditions that require substantial separation of the rings in the Mather spectrum.

As the results may suggest, the methods employed in proving these theorems are quite different from the robust Hadamard method, which underlies the construction of stable manifolds (for which the Perron–Irwin method could also be used) as well as the study of transverse regularity.

h. Local product structure. Consider a hyperbolic set $\Lambda \subset M$ for an embedding $f\colon U \to M$. If for $x \in \Lambda$ and $\epsilon > 0$ we set $W^s_\epsilon(x) := \{y \in M \mid d(f^n(x), f^n(y)) < \epsilon \text{ for } n \in \mathbb{N}\}$ and $W^u_\epsilon(x) := \{y \in M \mid d(f^{-n}(x), f^{-n}(y)) < \epsilon \text{ for } n \in \mathbb{N}\}$, then for sufficiently small ϵ a map $[\cdot,\cdot]\colon \Lambda \times \Lambda \to M$ is well-defined by setting $W^u_\epsilon(x) \cap W^s_\epsilon(y) = \{[x,y]\}$. We say that Λ has *local product structure* if $[x,y] \in \Lambda$ for $x, y \in \Lambda$. This is equivalent to local maximality.

i. Stable and unstable manifolds for flows. Let Λ be a (λ,μ)-hyperbolic set for a C^r flow $\varphi^t\colon M \to M$, $r \in \mathbb{N}$, λ, μ as in the definition, and $t_0 > 0$. Then for each $x \in \Lambda$ there is a pair of embedded C^r-discs $\widetilde{W}^s(x)$, $\widetilde{W}^u(x)$, called the *local strong stable manifold* and the *local strong unstable manifold* of x, respectively, such that

1. $T_x\widetilde{W}^s(x) = E^-_x$, $T_x\widetilde{W}^u(x) = E^+_x$;
2. $\varphi^t(\widetilde{W}^s(x)) \subset \widetilde{W}^s(\varphi^t(x))$ and $\varphi^{-t}(\widetilde{W}^u(x)) \subset \widetilde{W}^u(\varphi^{-t}(x))$ for $t \geq t_0$;
3. for every $\delta > 0$ there exists $C(\delta)$ such that
$$d(\varphi^t(x), \varphi^t(y)) < C(\delta)(\lambda + \delta)^t d(x, y) \qquad \text{for } y \in \widetilde{W}^s(x),\ t > 0,$$
$$d(\varphi^{-t}(x), \varphi^{-t}(y)) < C(\delta)(\mu - \delta)^{-t} d(x, y) \quad \text{for } y \in \widetilde{W}^u(x),\ t > 0;$$
4. there exists a continuous family U_x of neighborhoods of $x \in \Lambda$ such that
$$\widetilde{W}^s(x) = \{y \mid \varphi^t(y) \in U_{\varphi^t(x)}, \quad t > 0, \quad d(\varphi^t(x), \varphi^t(y)) \xrightarrow[t\to\infty]{} 0\},$$
$$\widetilde{W}^u(x) = \{y \mid \varphi^{-t}(y) \in U_{\varphi^{-t}(x)}, t > 0, \quad d(\varphi^{-t}(x), \varphi^{-t}(y)) \xrightarrow[t\to\infty]{} 0\}.$$
5. Global *strong stable* and *strong unstable* manifolds
$$W^s(x) := \bigcup_{t>0} \varphi^{-t}(\widetilde{W}^s(\varphi^t(x))) \text{ and } W^u(x) := \bigcup_{t>0} \varphi^{t}(\widetilde{W}^u(\varphi^{-t}(x)))$$
are well-defined, are smooth injectively immersed, and are characterized by
$$W^s(x) = \{y \in M \mid d(\varphi^t(x), \varphi^t(y)) \xrightarrow[t\to\infty]{} 0\},$$
$$W^u(x) = \{y \in M \mid d(\varphi^{-t}(x), \varphi^{-t}(y)) \xrightarrow[t\to\infty]{} 0\}.$$

With a little care one can replace the condition $t \geq t_0$ in 2. by $t > 0$.

The manifolds $W^{0s}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^s(x))$ and $W^{0u}(x) := \bigcup_{t\in\mathbb{R}} \varphi^t(W^u(x))$ are called the *weak stable* and *weak unstable* manifolds of x. They form Hölder continuous laminations with smooth leaves, as do the strong stable and unstable leaves [**A2**]. Note that $T_xW^{0s} = E^0_x \oplus E^-_x$ and $T_xW^{0u} = E^0_x \oplus E^+_x$.

3. Regularity of the invariant laminations

a. Definitions. To discuss the regularity of these laminations (as opposed to that of the leaves) in any detail we begin with definitions. Given $\alpha \geq 0$ we say that a map is C^α if its derivatives of order $\lfloor\alpha\rfloor$ are $\alpha - \lfloor\alpha\rfloor$-Hölder continuous (where 0-Hölder means just continuous). By $C^{k,\omega}$ we denote those C^k maps whose kth derivatives have modulus of continuity ω. These regularity classes are the appropriate ones for the invariant laminations, but we first need to describe how to define their regularity.

The regularity of subbundles is unambiguously defined, *e.g.,* through the optimal regularity of spanning vector fields in smooth coordinate systems. For the dependence of the leaves on the base point several slightly different definitions are possible. The canonical definition is via the highest possible regularity of lamination charts. One may also look into the transverse regularity of k-jets. Alternatively, one can examine the holonomy semigroup, *i.e.,* for pairs of nearby smooth transversals to the lamination one considers the locally defined map between them that is obtained by "following the leaves". By transversality this is well-defined, and for smooth transversals one can discuss the regularity of these maps, which turns out to be largely independent of the transversals chosen. This is what we will study, and we refer to it as the regularity of holonomies or (transverse) regularity of the lamination. There is little difference between these definitions in our context. Following the discussion in [**PSW**] we can summarize the relation as follows:

THEOREM 2.3.1 ([**PSW**, Theorem 6.1]). *If $r \in \mathbb{R}\cup\{\infty\}$, $r \notin \mathbb{N}\setminus\{1\}$ then a foliation with uniformly C^r leaves and holonomies has C^r foliation charts.*

However, if $r \in \mathbb{N} \smallsetminus \{1\}$ then a foliation with uniformly C^r leaves and holonomies need not have C^r foliation charts. The problem are mixed partials. Without assuming uniform regularity the above statements can fail drastically: There is a foliation with uniformly C^∞ leaves and with (nonuniformly) C^∞ holonomies that does not have a C^1 foliation chart [**PSW**, Figure 9]. In our context the regularity is always uniform, so the above result implies that one can define regularity equally well via holonomies or foliation charts. The essential ingredient for Theorem 2.3.1 is Journé's Theorem 2.3.9, by the way. This leads to the following observation.

THEOREM 2.3.2. *If $r \in \mathbb{R} \cup \{\infty\}$, $r \notin \mathbb{N} \smallsetminus \{1\}$ and the stable and unstable foliations have uniformly C^r holonomies, then there are C^r* bifoliation charts, i.e., *charts that straighten both foliations simultaneusly.*

PROOF. The hypothesis implies that every point p has a neighborhood U on which the inverse $[x,y] \mapsto (x,y) \in W^u_\epsilon(p) \times W^s_\epsilon(p)$ of the local product structure map (Subsection 2.2h) is uniformly C^r in either entry. By Theorem 2.3.9 it is C^r. □

There is a connection between the regularity of the subbundles and that of the lamination: For any $r \in \mathbb{N} \cup \{\infty\}$ and $\alpha \in [0,1)$ or "$\alpha = \text{Lip}$" a foliation tangent to a $C^{r+\alpha}$ subbundle is itself $C^{r+\alpha}$ [**PSW**, Table 1]. (The reverse implication holds only for $r = \infty$ because leaves tangent to a C^r subbundle are C^{r+1}.)

The invariant subbundles are always Hölder continuous (Proposition 2.3.3). It should be noted, however, that for $\alpha < 1$ the α-Hölder condition on subbundles does not imply any regularity of the foliations. Indeed, without a Lipschitz condition even a one-dimensional subbundle may not be uniquely integrable, so already continuity of the foliation cannot be obtained this way. On the other hand, there turns out to be a recently discovered converse connection: If the holonomies are α-Hölder and individual leaves are C^∞ then the subbundles are β-Hölder for every $\beta < \alpha$ [**HW**]. (There are variants of this for leaves of finite smoothness and almost-everywhere Hölder conditions.) Furthermore, whenever bunching-type information gives a particular degree of regularity for the subbundles, one can usually get the same regularity for the holonomies, and vice versa.

b. Hölder regularity. As mentioned earlier, the holonomies are always Hölder. This was first used to obtain absolute continuity of holonomies as the base for a proof of ergodicity by the Hopf argument (Subsection 2.3h). One can give a lower bound for the optimal Hölder exponent in terms of contraction and expansion rates. Classical sources for this are [**HPS, Fn1**].

The relevant information about contraction and expansion rates is given by the *bunching parameter* $B^u(f) := \sup\{\inf_{p\in M}(\log\mu_s(p) - \log\nu_s(p))/\log\mu_f(p) \mid \mu_f(p) < \mu_s(p) < 1 < \nu_s(p),\ \mu_f^n(p)\|v\|/C \le \|Df^n(v)\| \le C\mu_s^n(p)\|v\|$ and $\|Df^{-n}(u)\| \le C\nu_s^{-n}(p)\|u\|$ for $v \in E^s(p),\ u \in E^u(p),\ n \in \mathbb{N}\}$. This is always positive by hyperbolicity and compactness. For example, in the symplectic case $\nu_s(p)\mu_s(p) = 1$, so $B^u(f) = 2\sup\inf_p \log\mu_s(p)/\log\mu_f(p)$ is close to 2 if and only if the contraction rates $\mu_s(p)$ and $\mu_f(p)$ are close together.

PROPOSITION 2.3.3. *If $B^u(f) \notin \mathbb{N}$ then $E^u \in C^{B^u(f)}$ and $W^u \in C^{B^u(f)}$; if $B^u(f) \in \mathbb{N}$ then $E^u \in C^{B^u(f)-1,O(x|\log x|)}$.*

Precursors of this result go a long way back [**HPS**]. For E^u this is due to Hasselblatt [**Hb5**] in this form, for the holonomies this is a consequence if $B^u(f) > 1$; for the holonomies and $B^u(f) < 1$ this is due to Schmeling–Siegmund-Schultze [**SS**], where essentially Lyapunov exponents are used, see also [**PSW**]. One should note that $B^u(f)$ could be quite large, but $B^u(f)$ and (the analogously defined) $B^s(f)$ cannot simultaneusly exceed 2, so this result together with its stable counterpart never claims that both laminations are C^2. This result also applies to the transverse regularity of k-jets. Note that the definition of the bunching parameter above involves rates of ratios, as it were, as does the work of Fenichel [**Fn1**]. Hirsch, Pugh and Shub [**HPS**], on the other hand, use a ratio of rates instead, which is a more stringent assumption (this is compared in [**Fn1**]), although they remark [**HPS**, Remark 1, p. 38] that their methods would yield analogous results for rates of ratios.

Two special cases are worth mentioning:

COROLLARY 2.3.4. 1. *If E^u has codimension one then there is only one contraction rate and $B^u(f) > 1$, hence E^u and W^u are $C^{1+\epsilon}$.*
2. *If, in addition, volume is preserved one also finds the same for the stable subbundle and lamination.*

This is one of the places where the smoothness of the diffeomorphism matters; for C^1 diffeomorphisms these statements fail badly [**PPR**].

c. Hyperbolic sets. The basic results from above apply essentially verbatim to the situation of a hyperbolic set other than the entire manifold. Notably Proposition 2.3.3 applies in that situation (the proof in [**Hb3, Hb5**] extends to that case). This observation can be strengthened in a remarkable way. Outside of the Anosov situation the "foliations" only fill up part of the ambient manifold (locally they are the product of a Cantor set with a disk), and therefore it is natural to ask whether they can be "filled in" or extended to nice foliations. McSwiggen [**McS2**] has shown that for a hyperbolic set where Proposition 2.3.3 would apply to give regularity higher than C^1 there is, in fact, a jet along the foliations of corresponding regularity. (In the Anosov case this just encodes the existing regularity.) In other words, one can "infinitesimally extend" the foliations, and with the same regularity. Indeed, McSwiggen expects to show that there is an actual extension with the same regularity. Even without the regularity conclusion this is nontrivial because not every continuous lamination can be filled in continuously [**McS1**].

One can strengthen Proposition 2.3.3 in an altogether different direction as well: For fractal hyperbolic sets (*i.e.,* hyperbolic sets other than the Anosov case) the holonomies appear to be Lipschitz-continuous off a subset of smaller fractal dimension. This has been proved in special cases [**HS**] with a method that appears general enough, and this should be useful for dimension calculations [**B**]. This is in contrast to the situation with symplectic Anosov diffeomorphisms (Theorem 2.3.5).

d. Obstructions to higher regularity. There might be infinitesimal improvements to the above regularity result, but it is substantially optimal in a strong sense. Anosov noted that the invariant foliations are generically not C^2 for area-preserving Anosov diffeomorphisms; his idea underlies parts of Theorem 4.2.1 and is described in Subsection 4.2a, where we discuss rigidity results related to unusually high smoothness of the invariant foliations. Anosov also gave a volume-preserving example where the subbundles are almost nowhere C^1. In that example the optimal Hölder exponent is 2/3 almost everywhere (with respect to volume) [**A1**]. One can make much stronger optimality assertions:

THEOREM 2.3.5. *For an open dense set of* symplectic *Anosov systems the regularity predicted by computing B^u only from periodic points is not exceeded (*i.e., *if the rates compare badly at a single periodic point then the regularity is correspondingly low—at that periodic point)* [**Hb3**]. *An open dense set of Riemannian metrics do not have $C^{1+\text{Lip}}$ horospheric foliations* [**Hb3**].

Furthermore, for any $\epsilon > 0$ there is an open set of symplectic Anosov diffeomorphisms for which the subbundles and holonomies are C^ϵ at most on a (Lebesgue) null set [**HW**].

This symplectic Anosov situation is the most interesting, because this is a natural context where getting Lipschitz holonomies on a sufficiently large set would bypass the proof of absolute continuity of the foliations that is needed to get ergodicity of volume. At the same time this context is rather complementary to that of fractal hyperbolic sets as it was described in the previous subsection.

The distinction is essentially due to the fact that in the Anosov situation any fast stable leaf (if it exists at all) trivially lies in the hyperbolic set.

e. Geodesic flows. When one considers geodesic flows one may obtain bunching information from *curvature pinching* (Subsubsection 2.1f1). To that end the sectional curvature of a compact negatively curved Riemannian manifold N is said to be *relatively a-pinched* if $C \leq$ sectional curvature $< aC$ for some $C\colon N \to -\mathbb{R}_+$. If C is constant, the curvature is said to be (absolutely) a-pinched. It is not known to which extent these notions can actually differ in examples. A classical result is the following one, which is not easily found in this form in the literature.

THEOREM 2.3.6. *If $a \in (0,1)$ and the curvature is a-pinched then the invariant laminations are $C^{2\sqrt{a}}$.*

The best-known case is $a = 1/4$, giving C^1 laminations [**HP**], [**Kl2**, Theorem 3.2.17], and the cited proofs imply this result. A remarkable extension to weak 1/4-pinching is the following:

THEOREM 2.3.7 ([**Hs**]). *For a Riemannian manifold with curvature in $[-4,-1]$ the Anosov splitting is differentiable a.e. with respect to every ergodic invariant Borel probability measure on the unit tangent bundle.*

Absolute pinching directly controls the Mather spectrum (Subsection 2.1b), which is much stronger than the control required to obtain large values of B^u. Indeed, it suffices to assume relative pinching:

PROPOSITION 2.3.8. *For $a \in (0,1)$ a compact relatively a-pinched Riemannian manifold has C^{2a} horospheric laminations* [**Hb4**].

Note that gives lower regularity than Theorem 2.3.6, but covers the same range of exponents.

In the case of geodesic flows stable and unstable foliations are defined even in the nonpositively curved case, whether or not the geodesic flow is Anosov (horospheric foliations [**S-K**]). Therefore, pertinent regularity questions arise if the curvature is allowed to be zero in some places. For surfaces the resulting foliations are Hölder with $C^{1+\mathrm{Lip}}$ leaves under some minimal assumptions on flatness (the leaves are always $C^{3/2}$) [**GW**]. That in this setting the horospheres may not be C^3 was shown by Ballmann, Brin and Burns [**BBB**]. Even for surfaces (where negative curvature gives $C^{1+\mathrm{Zygmund}}$ holonomies) one may get non-Lipschitz holonomies (in higher dimension the Lipschitz property can fail even in negative curvature [**Hb3**]). If the curvature of a compact surface is negative except along a closed geodesic then the horocycle foliation may fail to be 1/2-Hölder at the corresponding orbit [**GN**] (this applies to subbundles as well as holonomies). However, the actual Hölder exponent is arbitrarily near 1 (without being 1) if the curvature vanishes to sufficiently high order at the closed geodesic [**GN**]. While the latter result holds in noncompact situations under some boundedness assumptions on derivatives of the curvature, without these the holonomies may fail to be Hölder altogether even if the curvature is negative and pinched and the volume is finite [**BBB**].

f. Bootstrap and rigidity. A somewhat complementary phenomenon is of some interest in regard to questions relating to high smoothness of the invariant foliations: There is an $N \in \mathbb{N}$ depending on contraction and expansion rates, such that if $E^u \in C^N$ then $E^u \in C^\infty$ (bootstrap, [**Hb1**]). For instance, in the case of sufficiently small perturbations of geodesic flows on constantly curved manifolds we have $N = 3$; as we see later (Theorem 4.2.7) this implies that no such perturbation can have C^3 invariant foliations (except for isometric metrics). Put differently, constant curvature metrics are rigid in the category of Riemannian metrics whose geodesic flows have C^3 horospheric foliations. This bootstrap was substantially refined by Foulon and Labourie [**FL**], who showed that sufficiently high regularity of a cocycle at a single periodic point forces high regularity everywhere.

g. Fast leaves. By work of Brin, Kifer [**BrK**] and Pesin [**BrP**] the subbundles E^i in Subsection 2.2f are Hölder continuous with exponent α, say, and by a result of Jiang, Llave, Pesin [**JLP**] the E^i are α-Hölder as a function of $g \in U$.

Examples of this situation are given by perturbations of higher-dimensional hyperbolic toral automorphisms and of geodesic flows of nonconstantly curved locally symmetric spaces.

An interesting observation is that even though transversely the fast laminations are usually only Hölder continuous, the fast leaves defined by the Lyapunov decomposition (Subsection 5.4b) C^1-laminate the next larger fast leaf of that decomposition [**LY, BPS**]. This is not implausible because of the similarity to the codimension one situation described above, where the foliation is C^1. Combining this with transverse differentiability or Lipschitz continuity (Subsection 2.3b) is useful for calculations of the fractal dimension of hyperbolic sets.

h. Absolute continuity and ergodicity of Anosov systems. Picking up Hopf's work on ergodicity of geodesic flows on compact negative curved surfaces, Anosov and Sinai [**A1, AS**] noted that the Hopf argument can be carried out in higher dimension because the invariant foliations, while not necessarily C^1, have absolutely continuous holonomy maps between nearby leaves [**S-HK**]. (This property requires the Anosov system to be a little better than C^1, see [**RoY**].) A contemporary rendering of the arguments is in [**Br3, BP**].

For proving ergodicity of volume in uniformly hyperbolic systems, absolute continuity can be bypassed by noting that volume, if invariant, is an equilibrium state (for the logarithm of the unstable Jacobian, Subsection 3.6e) and hence mixing (Subsection 3.6c), which implies ergodicity [**KH**].

i. Leafwise regularity. There are several situations where the regularity of some map is most easily seen to be high on each leaf of the invariant foliations. Although the leaves do not necessarily depend very smoothly on their base point one can nevertheless obtain global regularity from leafwise regularity. This was first observed by [**LMM1**] and [**HK**] and the most elementary proof combined with the strongest conclusion is due to Journé [**J**]:

THEOREM 2.3.9. *Let M be a C^∞ manifold, F^u, F^s continuous transverse foliations with uniformly smooth leaves, $n \in \mathbb{N}_0$, $\alpha > 0$, $f\colon M \to \mathbb{R}$ uniformly $C^{n+\alpha}$ on leaves of F^u and F^s. Then f is $C^{n+\alpha}$.*

CHAPTER 3

Topological dynamics, stability, invariant measures

1. Expansivity and local stability

a. Expansivity. The restriction of a diffeomorphism to a hyperbolic set is expansive: From 4. and 7. in Subsection 2.2e one sees that $\delta := \min(\beta, \epsilon)$ is an expansivity constant because if $d(f^n(x), f^n(y)) < \delta$, for $n \in \mathbb{Z}$, then $f^n(y) \in O_{f^n(x)}$ and $y \in \widetilde{W}^s(x) \cap \widetilde{W}^u(x) = \{x\}$.

This can also be proved via the hyperbolic fixed point theorem without using stable and unstable manifolds [**Yc**].

b. The Hartman–Grobman Theorem. Let M be a smooth manifold, $U \subset M$ open, $f\colon U \to M$ continuously differentiable, and $p \in U$ a hyperbolic fixed point of f. Then there exist neighborhoods U_1, U_2 of p, neighborhoods V_1, V_2 of $0 \in T_pM$ and a homeomorphism $h\colon U_1 \cup U_2 \to V_1 \cup V_2$ such that $f = h^{-1} \circ Df_p \circ h$ on U_1 [**Ht1, Gm, KH**], *i.e.,* the following diagram commutes:

$$\begin{array}{ccc} U_1 & \xrightarrow{f} & U_2 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ V_1 & \xrightarrow{Df_p} & V_2 \end{array}$$

A global version of the statement for Lipschitz-perturbations of linear maps is given and proved in [**S-HK, Yc, Pu2**]. The connection between the two versions is localization [**KH**, Lemma 6.2.7], [**Yc**, Section 2.4].

One can prove this result using stable and unstable manifolds or by obtaining the conjugacy via the hyperbolic fixed point theorem.

Thus, beyond the apparent similarity between the dynamics of a map near a hyperbolic fixed point and that of its linearization, these two are topologically conjugate. This conclusion can easily be strengthened.

c. Local topological rigidity. Since two invertible linear contractions with the same orientation are topologically conjugate, the Hartman–Grobman Theorem implies that the topological character of f near p is determined already by the orientation of f on stable and unstable manifolds and by their dimensions:

Suppose $f\colon U \to \mathbb{R}^n$, $g\colon V \to \mathbb{R}^n$ have hyperbolic fixed points $p \in U$ and $q \in V$, respectively, and for $i = +, -$

$$\dim E^i(Df_p) = \dim E^i(Dg_q), \quad \operatorname{sign}\det Df_p\restriction_{E^i(Df_p)} = \operatorname{sign}\det Dg_q\restriction_{E^i(Dg_q)}.$$

Then there exist neighborhoods $U_1 \subset U$ and $V_1 \subset V$ and a homeomorphism $h\colon U_1 \to V_1$ such that $h \circ f = g \circ h$.

Thus there is a complete finite set of invariants of local conjugacy. (Completeness of a set of invariants means that each set of values determines an equivalence class.) In particular any C^1 diffeomorphism is locally structurally stable in a neighborhood of a hyperbolic fixed point.

This result is a primitive precursor to the classification of Anosov diffeomorphisms in Subsection 3.5e.

At the same time there are improvements in another direction: The regularity of the linearizing homeomorphism is often better than only continuity. For example, if the manifold M in the Hartman–Grobman Theorem is two-dimensional then one can linearize by a C^1 local diffeomorphism [**Ht2**]. In general, the possibility of linearization of higher regularity is a matter of the presence or absence of resonances and is closely related to normal forms (Subsection 4.2a, Subsubsection 4.2b2).

2. Shadowing

Essentially by definition hyperbolic dynamical systems are distinguished by exponential divergence of any two orbits from each other. This constitutes the maximal sensitivity on initial conditions possible in smooth dynamics and is responsible for the complexity of the dynamics. A central aspect of hyperbolic dynamnics is, however, that this very instability of orbits coexists with and indeed causes a remarkable robustness of the orbit structure on the whole. The key connection between these distinct features is the shadowing property: "Approximate orbits" can always be approximated by genuine orbits. The basic technical device underlying this property, in turn, is again the Hyperbolic Fixed Point Theorem (or the Banach Contraction Principle).

a. Pseudo-orbits and shadowing. Let (X, d) be a metric space, $U \subset M$ open and $f\colon U \to X$. For $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z} \cup \{\infty\}$ a sequence $(x_n)_{a<n<b} \subset U$ is said to be an ϵ-*orbit* or ϵ-pseudo-orbit for f if $d(x_{n+1}, f(x_n)) < \epsilon$ for all $a < n < b$. It is said to be δ-*shadowed* by the orbit $\mathcal{O}(x)$ of $x \in U$ if $d(x_n, f^n(x)) < \delta$ for all $a < n < b$.

That the latter condition is nontrivial is illustrated by the simple example $f\colon S^1 \to S^1$, $x \mapsto x + 0.1 \cdot \sin^2 \pi x \pmod 1$ where all orbits are homoclinic to 0 but a pseudo-orbit can jump across 0.

b. The Anosov closing lemma. Let Λ be a hyperbolic set for $f\colon U \to M$. Then there exists an open neighborhood $V \supset \Lambda$ and C, $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$ and any periodic ϵ-orbit $(x_0, \dots, x_m) \subset V$ there is a point $y \in U$ such that $f^m(y) = y$ and $d(f^k(y), x_k) < C\epsilon$ for $k = 0, \dots, m-1$. In fact, $d(f^k(y), f^k(x)) < C\ \alpha^{\min(k,m-k)} \cdot (d(x,y) + d(f^m(x), f^m(y)))$ [**A1**].

This can be derived from the Hyperbolic Fixed Point Theorem in Subsection 2.1h [**Yc**] or from the Banach Contraction Principle [**KH**]. The Anosov Closing Lemma does not assert that the periodic orbit lies in Λ, but this is clearly true if Λ is a locally maximal hyperbolic set for $f\colon U \to M$. Thus, in this case periodic points are dense in $\Lambda' := NW(f\restriction_{\Lambda})$. In particular, periodic points are dense in a basic set.

One can prove a counterpart of the Anosov Closing Lemma for flows using Poincaré maps between successive transversals to a pseudo-orbit. Alternatively it follows from the Shadowing Theorem for flows (Subsection 3.2d).

c. Shadowing Lemma. Let M be a Riemannian manifold, $U \subset M$ open, $f\colon U \to M$ a diffeomorphism, and $\Lambda \subset U$ a compact hyperbolic set for f. Then there exists a neighborhood $U(\Lambda) \supset \Lambda$ such that whenever $\delta > 0$ there is an $\epsilon > 0$ so that every ϵ-orbit in $U(\Lambda)$ is δ-shadowed by an orbit of f.

The markedly sensitive dependence of an orbit on its initial point poses the problem of extracting meaningful information from approximate knowledge of an orbit segment. The Shadowing Lemma addresses this point and in particular helps ascertain that numerical calculations reflect actual orbits accurately. Hyperbolicity suggests that any initial error grows exponentially, but this result nevertheless guarantees that the computed orbit represents a genuine one with satisfactory accuracy.

However, shadowing as such does not guarantee that the numerical pseudo-orbit is in any sense *typical*: For the map $E_2\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto 2x \pmod 1$ any computer-generated orbit eventually becomes zero since the initial condition is internally represented by a binary fraction and at each step the number of nonzero binary digits after the point decreases. Thus the computer always computes an actual orbit, but always one attracted to zero. Since the circle is a hyperbolic repeller, this is highly untypical. Typical orbits are equidistributed with respect to Lebesgue measure (Subsection 3.6e).

d. Shadowing Theorem. A stronger version provides for coherent shadowing of entire continuous families of ϵ-orbits.

Let M be a Riemannian manifold, $U \subset M$ open, $f\colon U \to M$ a diffeomorphism, and $\Lambda \subset U$ a compact hyperbolic set for f.

Then there exist a neighborhood $U(\Lambda) \supset \Lambda$ and $\epsilon_0, \delta_0 > 0$ such that for all $\delta > 0$ there is an $\epsilon > 0$ with the following property:

If $f'\colon U(\Lambda) \to M$ is a C^2 diffeomorphism ϵ_0-close to f in the C^1 topology, Y a topological space, $g\colon Y \to Y$ a homeomorphism, $\alpha \in C^0(Y, U(\Lambda))$, and $d_{C^0}(\alpha g, f'\alpha) := \sup_{y \in Y} d(\alpha g(y), f'\alpha(y)) < \epsilon$ then there is a $\beta \in C^0(Y, U(\Lambda))$ such that $\beta g = f'\beta$ and $d_{C^0}(\alpha, \beta) < \delta$. Furthermore β is locally unique: If $\bar{\beta} g = f'\bar{\beta}$ and $d_{C^0}(\alpha, \bar{\beta}) < \delta_0$, then $\bar{\beta} = \beta$.

To see that this implies the Shadowing Lemma take $Y = \mathbb{Z}$, $f' = f$, $\epsilon_0 = 0$, and $g(n) = n + 1$ and replace $\alpha \in C^0(Y, U(\Lambda))$ by $\{x_n\}_{n \in \mathbb{Z}} \subset U(\Lambda)$ and "$\beta \in C^0(Y, U(\Lambda))$ such that $\beta g = f'\beta$" by $\{f^n(x)\}_{n \in \mathbb{Z}} \subset U(\Lambda)$. Then $d(x_n, f^n(x)) < \delta$ for all $n \in \mathbb{Z}$.

The Anosov Closing Lemma is obtained by taking $f' = f$, $Y = \mathbb{Z}/n\mathbb{Z}$, $g(k) = k + 1 \pmod n$.

This result can be proved by utilizing stable and unstable manifolds [**KH**], or by way of the hyperbolic fixed point theorem. In [**KH**] this result is used to prove structural stability.

e. The specification property. Let $f\colon X \to X$ be a bijection of a set X. A *specification* $S = (\tau, P)$ consists of a finite collection $\tau = \{I_1, \dots, I_m\}$ of finite intervals $I_i = [a_i, b_i] \subset \mathbb{Z}$ and a map $P\colon T(\tau) := \bigcup_{i=1}^m I_i \to X$ such that for $t_1, t_2 \in I \in \tau$ we have $f^{t_2}(P(t_1)) = f^{t_1}P(t_2)$. S is said to be n-*spaced* if $a_{i+1} > b_i + n$ for all $i \in \{1, \dots, m\}$ and the minimal such n is called the *spacing* of S. We say that S *parameterizes* the collection $\{P_I \mid I \in \tau\}$ of orbit segments of f.

We let $T(S) := T(\tau)$ and $L(S) := L(\tau) := b_m - a_1$. If (X, d) is a metric space we say that S is ϵ-*shadowed* by $x \in X$ if $d(f^n(x), P(n)) < \epsilon$ for all $n \in T(S)$.

Thus a specification is a parameterized union of orbit segments $P\restriction_{I_i}$ of f.

If (X, d) is a metric space and $f\colon X \to X$ a homeomorphism then f is said to have the *specification property* if for any $\epsilon > 0$ there exists an $M = M_\epsilon \in \mathbb{N}$ such that any M-spaced specification S is ϵ-shadowed by a point of Λ and such that moreover for any $q \geq M + L(S)$ there is a period-q orbit ϵ-shadowing S. Thus, any finite collection of finite orbit segments can be embedded with arbitrary accuracy in a single (periodic) orbit.

Note that the specification property implies topological mixing. One can also show that expansivity and the specification property together imply positivity of topological entropy (when $\mathrm{card}(X) > 1$)[**KH**].

The specification property is mentioned here because it holds for topologically mixing hyperbolic sets (Bowen's specification theorem, Subsection 3.3a). Bowen developed a great deal of hyperbolic dynamics in the framework of expansive systems with the specification property, see *e.g.*, Subsection 3.4a.

f. Specification for flows. Suppose φ is a flow on a set X. A *specification* $S = (\tau, P)$ consists of a finite collection $\tau = \{I_1, \dots, I_m\}$ of bounded intervals $I_i = [a_i, b_i] \subset \mathbb{R}$ and a map $P\colon T(\tau) := \bigcup_{I\in\tau} I \to X$ such that for $t_1, t_2 \in I \in \tau$ we have $\varphi^{t_2}(P(t_1)) = \varphi^{t_1} P(t_2)$. S is said to be *θ-spaced* if $a_{i+1} > b_i + \theta$ for all $i \in \{1, \dots, m\}$ and the minimal such θ is called the *spacing* of S. We say that S *parameterizes* the collection $\{P_I \mid I \in \tau\}$ of orbit segments of f.

We let $T(S) := T(\tau)$ and $L(S) := L(\tau) := \max T(\tau) - \min T(\tau)$. If (X, d) is a metric space we say that S is *ϵ-shadowed* by $x \in X$ if $d(\varphi^t(x), P(t)) < \epsilon$ for all $t \in T(S)$.

If (X, d) is a metric space and φ a flow then φ is said to have the *specification property* if for any $\epsilon > 0$ there exists an $M = M_\epsilon \in \mathbb{R}$ such that any M-spaced specification S is ϵ-shadowed by a point of Λ and moreover such that for any $s \geq M + L(S)$ there is a period-s' orbit ϵ-shadowing S with $|s - s'| < \epsilon$.

g. Closing Theorems. In this subsection we describe further closing theorems. The Anosov Closing Lemma perturbs nonwandering points to periodic ones within the same dynamical system. In the absence of hyperbolicity this may be impossible, such as for an irrational rotation. However, even for issues related to hyperbolic dynamics it is important to have closing theorems that involve perturbations of the dynamical system to close a nonwandering or recurrent orbit.

There are two important such results that require no hyperbolicity. The Pugh Closing Lemma says that nonwandering orbits can be closed by a localized C^1 perturbation of the map [**Pu1**] and the Mañé Ergodic Closing Lemma makes an analogous assertion for almost every orbit with respect to any invariant Borel probability measure [**M1**]. Both are discussed in [**S-HK**]. See [**Ad**] for simpler proofs of these results and for further developments.

The Hayashi Connecting Lemma [**Hy**] builds on the previous two statements and belongs firmly into the hyperbolic context. Given a hyperbolic set Λ and $i = u, s$ let $W^i_\epsilon(\Lambda) = \bigcup_{x\in\Lambda} W^i_\epsilon(x)$. Set $D^s = \overline{W^s_\epsilon(\Lambda) - f(W^s_\epsilon(\Lambda))}$ and $D^u = \overline{W^u_\epsilon(\Lambda) - f^{-1}(W^u_\epsilon(\Lambda))}$. In case of a flow replace f by the time-one map. *Homoclinic* points of Λ are those in $(W^s(\Lambda) \cap W^u(\Lambda)) \smallsetminus \Lambda$. The theorem says that an almost homoclinic situation can be perturbed to a homoclinic one:

THEOREM 3.2.1 ([**Hy**]). *Let M be a compact manifold, f a differentiable dynamical system, U a C^1 neighborhood of f, Λ a locally maximal hyperbolic set. Suppose there is a sequence $(\gamma_i)_{i\in\mathbb{N}}$ of finite orbit segments that accumulates on both D^s and D^u and such that each γ_i starts in a neighborhood of D^u, leaves an isolating neighborhood of Λ (Subsection 2.1c) and then enters a neighborhood of D^s. Then there is a $g \in U$ coinciding with f in a neighborhood of Λ for which Λ has a homoclinic point.*

A simpler proof of a special case is in [**X**].

This lemma is the crucial device in the proof that structurally stable *flows* are hyperbolic (Subsection 3.5b), but has already proved useful in other contexts.

3. Transitivity

In general, topological or smooth dynamical systems cannot be canonically decomposed into topologically transitive pieces, unlike the ergodic decomposition in the measurable category. The complexity of the orbit structure in hyperbolic dynamical systems, however, essentially forces this topological irreducibility of the nonwandering set, modulo finite unions. Thus, hyperbolic dynamical systems admit a topological decomposition that is much more effective than an ergodic decomposition is guaranteed to be: Each hyperbolic dynamical system breaks up into a *finite* union of transitive pieces. The essential reason is that orbits are so intertwined as to produce "local transitivity"; compactness then implies finiteness of the decomposition.

a. Spectral decomposition. Let Λ be a locally maximal hyperbolic set for $f\colon U \to M$. Then $\Lambda' := NW(f\restriction_{\Lambda}) = \bigcup_{i=1}^{k} \Lambda_i$ with $\Lambda_i \subset \overline{W^u(x)}$, $\Lambda_i \subset \overline{W^s(x)}$ for $x \in \Lambda_i$, $f\restriction_{\Lambda_i}$ is topologically transitive and $\Lambda_i = \bigcup_{j=1}^{k_i} \Lambda_{ij}$ disjointly with $f(\Lambda_{ij}) = \Lambda_{i\,j+1}$ (where $\Lambda_{i\,k_i+1} := \Lambda_{i1}$) and $f^{k_i}\restriction_{\Lambda_{ij}}$ topologically mixing.

In particular regionally recurrent Anosov diffeomorphisms (those where $NW(f) = M$, see [**S-HK**]) are topologically mixing. Indeed, for Anosov diffeomorphisms the following are equivalent:

1. f is regionally recurrent
2. $W^u(x)$ is dense for all $x \in M$
3. $W^s(x)$ is dense for all $x \in M$
4. Periodic points are dense
5. f is topologically transitive
6. f is topologically mixing

Using stable and unstable manifolds as well as the Anosov Closing Lemma (Subsection 3.2b) one can show that any topologically mixing locally maximal hyperbolic set has the specification property (Subsection 3.2e). Together with expansivity this implies positive topological entropy (unless $\operatorname{card}\Lambda \le 1$) as well as existence of a unique invariant probability measure of maximal entropy, with respect to which periodic points are equidistributed. (Alternatively one can deduce all this from Markov partitions [**S-C, KH**], see Subsection 3.6b.)

Via the spectral decomposition this translates into existence of such measures for the nonwandering set of any compact locally maximal hyperbolic set plus uniqueness in the topologically transitive case [**KH**].

b. Spectral decomposition and mixing for flows. Let Λ be a locally maximal hyperbolic set for $\varphi^t\colon \mathbb{R} \times U \to M$. Then $\Lambda' := NW(\varphi^t\restriction_{\Lambda}) = \bigcup_{i=1}^{k} \Lambda_i$ with $\Lambda_i \subset \overline{W^{0u}(x)} \cap \overline{W^{0s}(x)}$ for $x \in \Lambda_i$ and $\varphi^t\restriction_{\Lambda_i}$ is topologically transitive. For each i we either have $\Lambda_i \subset \overline{W^u(x)} \cap \overline{W^s(x)}$ for $x \in \Lambda_i$ and $\varphi^t\restriction_{\Lambda_i}$ topologically mixing, or $\varphi^t\restriction_{\Lambda_i}$ is a special flow over a homeomorphism (which satisfies the hyperbolicity condition Axiom A* [**PeS**] and Axiom A$^\#$ [**AJ**]; it is Anosov if $\varphi^t\restriction_{\Lambda'}$ is).

The neat reduction to the topologically mixing case offered by the spectral decomposition in discrete time has no counterpart for flows. The reason is that suspensions are not mixing; they lack "mixing in the time direction" (Subsubsection 2.1f2). (In discrete time this is not an issue. For example, the spectral decomposition of a single periodic orbit is into singletons, i.e., time can be absorbed into the spectral decomposition).

The distinction arises from the difference between weak and strong (un)stable foliations for flows. The spectral decomposition is achieved by considerations of closures of (un)stable leaves. For flows this results in a decomposition into sets in which weak leaves are dense, which guarantees topological transitivity. However, topological mixing can only be obtained via density of *strong* (un)stable leaves, and this cannot be achieved by further invariant decomposition. The fundamental distinction here is via suspensions on the one hand, where strong leaves are coherently stacked and never dense, and, for example, geodesic (and contact) flows on the other hand, where the contact structure forces a complete nonintegrability (or *accessibility*) of the strong subbundles, which produces dense strong leaves and hence mixing. The point is that in the latter case traversing a small unstable–stable–unstable–stable quadrangle always results in a displacement in the flow direction, which causes the transverse mixing effects to produce "mixing in the flow direction" as well.

Indeed, we have the *Anosov alternative*: Transitive Anosov flows are either mixing or a suspension [**Pl1**] (in the volume-preserving case this is due to Anosov [**A1**]). To be more precise, if there is a nondense strong leaf or if $E^s \oplus E^u$ is integrable then the flow is a suspension. The idea of the proof is that the closure of a nondense strong leaf is a global section because every orbit intersects it by transitivity. It is smooth, and furthermore, the return time is constant.

By the way, transitivity and density of periodic points are both equivalent to density of weak stable leaves.

c. Transitivity of Anosov systems. The spectral decomposition implies that a regionally recurrent (every point is nonwandering, [**S-HK**]) Anosov system is transitive, and mixing in the discrete-time case. However, it is not known whether Anosov diffeomorphisms are always topologically transitive. All known examples are transitive [**Mn2**], and it is unknown whether there are any further examples. Newhouse [**Nh**] showed that codimension one Anosov diffeomorphisms (i.e., those for which one of the invariant subbundles is one-dimensional) are regionally recurrent.

On the other hand, it is known that Anosov flows need not be topologically transitive by virtue of an example [**FrW**] in dimension three. However, codimension one Anosov flows on manifolds of dimension greater than three are regionally recurrent and hence transitive [**Vj1**]. A related result is a condition on the Mather spectrum—that it is contained in two sufficiently thin annuli—that implies regional recurrence [**Br1**].

d. The Bowen–Ruelle alternative. At this point it is interesting to point out a basic dichotomy: Except for the Anosov situation, hyperbolic sets are always null sets. More precisely: A basic set (Subsection 2.1c) of a $C^{1+\alpha}$ diffeomorphism or flow either has Lebesgue measure zero or else is a connected component of the manifold [**BR**, Corollary 5.7]. Here is the proof, using Theorem 5.6 of [**BR**], which says that a basic hyperbolic set is an attractor if and only if the Lebesgue measure of the set of stable manifolds is positive. If $m(\Lambda) > 0$, where m is Lebesgue measure, then $m(W^s_\Lambda) > 0$ and $m(W^u_\Lambda) > 0$, so Λ is an attractor for the diffeomorphism and its inverse. The former implies $W^u_\Lambda = \Lambda$, the latter that W^u_Λ is open. Hence Λ is open (and closed). It is connected because the stable manifold of any periodic orbit is dense (the spectral decomposition is obtained from closures of stable manifolds).

One should note that this alternative depends on the regularity assumption: There is a horseshoe of positive Lebesgue measure for a C^1 diffeomorphism [**Bw2**].

4. Periodic points

Among the features of hyperbolic dynamics is the presence of periodic points of arbitrarily high period and their great abundance and rapid growth in number. The importance of periodic points is manifold. Poincaré recognized their utility as reference orbits where local analysis can help understand much more complicated nearby orbits. They themselves, their numbers and their growth rates provide useful conjugacy invariants. In hyperbolic dynamics their abundance also lends importance to the invariant measures supported on periodic orbits. These are used in the construction of important invariant Borel probability measures (Subsection 3.6c). Periodic points feature importantly in the survey [**S-FM**]. ζ-functions and related subjects are discussed in [**S-P**].

a. Exponential growth and entropy. The combination of expansivity and specification has various consequences that make up a good deal of the rich dynamics of symbolic systems and hyperbolic sets. These are outlined in [**S-HK**, Chapters 2 and 4] with more complete proofs and references in [**KH**]. Suppose $f\colon X \to X$ is a map of a compact metric space and for $\epsilon > 0$ and $m \in \mathbb{N}$ denote by $N(f, \epsilon, n)$ the maximal cardinality of a set $S \subset X$ such that $\epsilon \le \min_{x,y \in S} \max_{0 \le i \le n-1} d(f^i(x), f^i(y))$. Then

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0} \overline{\lim}_{n \to \infty} \frac{1}{n} \log N(f, \epsilon, n) = \lim_{\epsilon \to 0} \underline{\lim}_{n \to \infty} \frac{1}{n} \log N(f, \epsilon, n).$$

If f is expansive with expansivity constant δ then $P_n(f) \le N(f, \epsilon, n)$ for any $\epsilon < \delta$ and in particular $p(f) \le h_{\text{top}}(f)$. If f has the specification property then $P_n(f) \ge N(f, 2\epsilon, n - M_\epsilon)$; in particular $p(f) \ge h_{\text{top}}(f)$.

The proof simply uses that periodic points are δ-separated by expansivity and that the specification property produces separated sets consisting of periodic points. More subtle arguments yield a much better asymptotic if both properties are combined:

If X is a compact metric space and $f\colon X \to X$ an expansive homeomorphism with the specification property then $P_n(f) e^{-n h_{\text{top}}(f)}$ is bounded [**S-P, S-FM, S-HK**], [**KH**, Theorem 18.5.5]. Again, the argument is that expansivity gives a large number of distinguishable orbit segments of given length through a small ball, specification shows that any finite combination of these appears approximately in an actual (periodic) orbit.

An immediate consequence of the preceding result is that $h_{\text{top}}(f) > 0$ unless $\operatorname{card}(X) \le 1$ (by specification). Since Bowen showed that a mixing compact locally maximal hyperbolic set has the specification property, this means a nontrivial compact locally maximal hyperbolic set has positive topological entropy.

Bowen's construction of equilibrium states from periodic orbits [**S-HK, KH**] shows in yet another way that periodic orbits and the measures supported on them reflect essentially all the dynamical complexity. These parts of Bowen's work are presented in [**KH**], which also contains references to his papers.

b. The ζ-function. (See also [**S-P**].) The *dynamical ζ-function* of f defined by

$$\zeta_f(z) = \exp \sum_{n=1}^{\infty} \frac{P_n(f)}{n} z^n,$$

where z is a complex number, has radius of convergence $e^{-h_{\text{top}}(f)} > 0$. Because the coefficients are nonnegative there is a pole at $e^{-h_{\text{top}}(f)}$. This provides a novel means for studying entropy and, for the case of flows, has been used to show that topological entropy varies smoothly with the dynamical system [**KKPW**].

A beautiful result is that in the present setting this is always a rational function [**Mn1, Fd, S-FM, S-P**].

c. Fine growth asymptotic. While the preceding asymptotic for periodic orbit growth is already rather remarkable it can be substantially improved using Markov partitions, which are introduced in [**S-C**]. These describe a compact locally maximal hyperbolic set as a factor of a topological Markov chain, where the factor map is an almost-conjugacy in that it is almost invertible. In particular, one can arrange for (essential) bijectivity on periodic points. Therefore the asymptotic growth of periodic points coincides with that of a topological Markov chain, which can be calculated rather precisely in the transitive case. For some $\epsilon > 0$ one obtains that $|P_n(f)e^{-nh_{\text{top}}(f)} - 1| < Ce^{-\epsilon n}$.

d. Periodic orbit growth for flows. Combining expansivity and specification for flows yields periodic orbit asymptotics much like before: Let X be a compact metric space and Φ an expansive flow with the specification property. Then $tP_{t,\epsilon}(\Phi)e^{-th_{\text{top}}(\Phi)}$ is bounded, where $P_{t,\epsilon}$ denotes the number of orbits of period $\tau \in (t-\epsilon, t+\epsilon)$. The factor of t is related to the fact that separated sets are also "stacked" in the flow direction.

As before this applies to mixing hyperbolic sets and more subtle arguments give sharper results. The coincidence of the Bowen measure of maximal entropy (Subsection 3.6c) and the Margulis measure (characterized by uniform expansion on unstable leaves, see Subsection 3.6d, [**KH**]) implies that periodic points are equidistributed with respect to Margulis measure. Toll used this to show the following result [**T**], [**KH**, Section 20.6]:

Suppose $\Phi\colon \mathbb{R} \times M \to M$ is a topologically mixing Anosov flow on a compact Riemannian manifold M with $P_t(\Phi)$ periodic orbits of period at most t. Then

$$\lim_{t\to\infty} th_{\text{top}}(\Phi)P_t(\Phi)e^{-th_{\text{top}}(\Phi)} = 1, \textit{ i.e.}, P_t(\Phi) \sim \frac{e^{th_{\text{top}}(\Phi)}}{th_{\text{top}}(\Phi)}.$$

This was further improved by Dolgopyat [**DP, S-P**]: For geodesic flows Φ on smooth negatively curved compact surfaces and sufficiently strongly pinched negatively curved manifolds

$$P_t(\Phi) = \int_2^{te^{h_{\text{top}}(\Phi)}} \frac{1}{\log u}\, du + O(e^{t(h_{\text{top}}(\Phi)-\epsilon)}) \text{ as } t \to +\infty.$$

Because of the analogies to the prime number theorem (about asymptotic density of primes), results such as these are referred to as prime geodesic theorems.

5. Stability and classification

Various notions of structural stability are presented in [**S-HK**]. It was noted there that for hyperbolic dynamical systems the natural such notion is strong C^1 structural stability. It means that a dynamical system is topologically equivalent (conjugate or orbit equivalent, respectively) to any C^1 perturbation via a homeomorphism close to the identity. While smooth conjugacy and rigidity are important and active subjects (see Chapter 4), the fundamental structural theory of hyperbolic dynamics involves (Hölder) continuous conjugacies.

a. Strong structural stability of hyperbolic sets. Let $\Lambda \subset M$ be a hyperbolic set of the diffeomorphism $f\colon U \to M$. Then for any open neighborhood $V \subset U$ of Λ and every $\delta > 0$ there exists $\epsilon > 0$ such that if $f'\colon U \to M$ and $d_{C^1}(f\restriction_V, f') < \epsilon$ there is a hyperbolic set $\Lambda' = f'(\Lambda') \subset V$ for f' and a homeomorphism $h\colon \Lambda' \to \Lambda$ with $d_{C^0}(\mathrm{Id}, h) + d_{C^0}(\mathrm{Id}, h^{-1}) < \delta$ such that $h \circ f'\restriction_{\Lambda'} = f\restriction_\Lambda \circ h$. Moreover, h is unique when δ is an expansivity constant.

This result can be obtained fairly directly from the Shadowing Theorem [**KH**], or from the hyperbolic fixed point theorem combined with localization and an intermediate step, but without the use of stable/unstable manifolds [**Yc**]. Among the original sources is [**A1**], and Moser cast the proof in terms of solving functional equations [**Mos2, Mt1**]. Strong structural stability is one of the central features of hyperbolic sets and one of the motivations for studying hyperbolic dynamical systems. The corresponding result for flows produces an orbit equivalence. Either way the conclusion is that under C^1 perturbations one cannot deform the topological orbit structure; hyperbolic systems are *topologically rigid*. (This is sometimes also referred to as "hyperbolic continuation".)

For Anosov diffeomorphisms there is a related result of Walters [**Wt1**]: A C^0 perturbation of an Anosov diffeomorphism f has f as a factor (topological stability). Moreover, if the perturbation has a sufficiently large expansivity constant then the factor map is a homeomorphism. This means that Anosov diffeomorphisms sufficiently C^0 close to f are topologically conjugate to f, *i.e.,* Anosov diffeomorphisms are C^0 structurally stable within the class of Anosov diffeomorphisms. The result of Walters was extended by Nitecki [**N2**], who showed that Smale's sufficient conditions for Ω-stability (Subsection 3.5c, [**S5**]) suffice for topological Ω-stability, and that Robbin's sufficient conditions (Subsection 3.5b, [**Rb**]) for structural stability suffice for topological stability.

The conjugacy is always Hölder with Hölder inverse. In dimension two one can also show [**PaV**] that the Hölder exponent of the conjugacy as a function of the C^1-perturbation is continuous at the identity, *i.e.,* is as close to 1 as desired if the perturbation is sufficiently C^1-small. The same can be achieved assuming conformal behavior (only one contraction/expansion rate) in the stable/unstable directions. This, combined with continuous dependence of the conjugacy on the perturbation suggests that the Hölder exponent of the conjugacy should in general tend to 1 as the perturbation disappears. Unfortunately, this is far from true [**HW**]: For any $\epsilon > 0$ there are linear Anosov diffeomorphisms conjugate to arbitrarily small perturbations via a conjugacy that is almost nowhere bi-C^ϵ (*i.e.,* almost nowhere are h and h^{-1} both C^ϵ). The specific ϵ depends on the disparity in the expansion or contraction rates of the linear map. The proof uses that when these disparities are great, such as for the matrix $A_\epsilon := \begin{pmatrix} B & 0 \\ 0 & B^{\lfloor 2/\epsilon^2 \rfloor + 1} \end{pmatrix}$, where $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then for any small perturbation the bunching constant is quite small and for symplectic perturbations the invariant foliations can be arranged to have low Hölder exponent (Theorem 2.3.5). Since the invariant structures for the linear system are analytic, the conjugacy cannot be very regular.

b. Strong transversality and the Stability Theorem. A subtle defect of the preceding result is that it refers to structural stability of the restriction to the hyperbolic set—without further conditions there is no guarantee that the dynamics of f off the hyperbolic set has any similarity to that of perturbations. In other words, this is not full structural stability and additional conditions away from the hyperbolic set are required in order to include the "dissipative" or wandering part.

A sufficient condition for structural stability can be expressed in terms of stable and unstable manifolds. According to a result of Smale [**S4**] $M = \bigcup_{x \in NW(f)} W^u(x) =$

$\bigcup_{x\in NW(f)} W^s(x)$ for an Axiom A diffeomorphism (Subsection 2.1c), which implies that every $x \in M$ has injectively immersed smooth stable and unstable manifold. The *strong transversality condition* is that $T_xW^s(x)+T_xW^u(x) = T_xM$ for every $x \in M$, *i.e.*, stable and unstable manifolds are in general position. Note that the sum may not be direct, *e.g.*, for a circle map with one attractor and one repeller, such as $f\colon S^1 \to S^1$, $x \mapsto x+0.1\cdot\sin 2\pi x \pmod 1$. That a condition of this kind is needed for structural stability arises from the observation that a tangency between a stable and an unstable manifold could be radically affected by a perturbation, causing a bifurcation [**PaT**].

One of the high points of smooth dynamics is the stability theorem:

THEOREM 3.5.1. *C^1 structural stability is equivalent to Axiom A together with strong transversality.*

This was conjectured by Palis and Smale [**PaS**] in 1968 and sufficiency of Axiom A and strong transversality for C^2 structural stability was proved by Robbin [**Rb**] and then refined to C^1 by Robinson [**Ro3**], who also showed that stability and Axiom A together imply strong transversality [**Ro1**]. In 1986 Mañé achieved the proof of necessity [**M3**]. This paper built on his development of numerous intricate technical tools, such as the Ergodic Closing Lemma (Subsection 3.2g, [**S-HK, M1**]).

Robbin's proof of structural stability for an Axiom A diffeomorphism f with strong transversality goes as follows. The hyperbolic splitting on the components of the spectral decomposition (Subsection 3.3a) is extended to an invariant neighborhood. Strong transversality allows some consistency in doing this. This is technically the most difficult part. To find a conjugacy to a nearby diffeomorphism g pass to C^1 vector fields by setting $f^{-1}\circ g = \exp(\xi)$ and looking for a conjugacy $h = \exp(\eta)$. Then $g\circ h = h\circ f$ becomes $(1-f_*)\eta = R_\xi(\eta)$, where f_* is the action on C^0 vector fields and R_ξ is a strong contraction for small ξ. In the Anosov case one can invert $1-f_*$ to get a unique $\eta = (1-f_*)^{-1}R_\xi(\eta)$ by the Banach Contraction Principle. In the present situation one instead looks for a right inverse J of $1-f_*$ and then takes the unique $\eta = JR_\xi(\eta)$. But one needs J such that $\exp(\eta)$ is injective. This holds if either η has a small Lipschitz constant or if f is expansive. Each of these is too restrictive, but these ideas can be combined by requiring η to have small Lipschitz constant with respect to the metric $d_f(x,y) := \sup_{n\in\mathbb{Z}} d(f^n(x), f^n(y))$. This is achieved by making J continuous on d_f-Lipschitz vector fields as follows:

$$J(\theta)(x) := \begin{cases} \sum_{n=0}^{\infty} Df^{-n}\circ\theta\circ f^n(x) & \text{if } \theta \text{ is a stable vector field} \\ -\sum_{n=1}^{\infty} Df^{n}\circ\theta\circ f^{-n}(x) & \text{if } \theta \text{ is an unstable vector field} \end{cases}$$

makes sense locally and can be pieced together globally due to the consistent extension of the invariant laminations at the start.

Mañé's proof of the converse builds on extensive earlier work by himself and others. As mentioned above, it was enough to prove Axiom A [**Ro1**]. Hyperbolicity of periodic points and their density in the nonwandering set had already been achieved, but uniform hyperbolicity of the nonwandering set was still to be shown. To this end Mañé proved that the closure of the set P_i of periodic points with i-dimensional stable manifold is a hyperbolic set. The argument uses induction, starting from the previously known case $i = 0$. Structural stability enters in showing that for $j \neq i$ we have $\overline{P_i}\cap\overline{P_j} \neq \varnothing$: Otherwise one can construct a perturbation with a new homoclinic point.

c. The Ω-stability theorem. A diffeomorphism $f\colon M \to M$ is said to be *C^1-Ω-stable* if there exists an $\epsilon > 0$ such that whenever $d_{C^1}(f,g) < \epsilon$ then $f\restriction_{NW(f)}$ and

$g\restriction_{NW(g)}$ are topologically conjugate. (The terminology derives from the historically popular usage $\Omega(f) := NW(f)$.)

This notion is reminiscent of structural stability of hyperbolic sets, however, even if $NW(f)$ is assumed hyperbolic it is a stronger requirement because the assertion implies that the nonwandering set of a perturbation is no larger (no "Ω-explosions" [**N1**]). Thus it entails control of the nonwandering set plus topological conjugacy. The point of such a notion is that interesting asymptotic behavior, in particular recurrence, is concentrated on the nonwandering set, and hence stability of the dynamics on it is what is truly essential for maintaining all asymptotic aspects of the recurrent orbit structure ($NW(f)$ contains all α- and ω-limit sets).

Hyperbolicity is an important ingredient for Ω-stability but again not sufficient by itself. If $NW(f)$ is hyperbolic let Λ_i denote the elements of the spectral decomposition. Then a *cycle* on $NW(f)$ is a pair of periodic sequences $x^l_j \in \Lambda_{i_j}$ $(l = 1, 2)$ with $W^s(x^1_j) \cap W^u(x^2_{j+1}) \neq \varnothing$. A cycle for an Axiom A diffeomorphism produces C^r perturbations with an extra periodic orbit [**Pa1**], so these need to be ruled out. One obtains the *Ω-stability theorem*:

THEOREM 3.5.2. *A diffeomorphism is Ω-stable if and only if it satisfies Axiom A and has no cycles.*

This was conjectured by Palis and Smale [**PaS**] when Smale proved sufficiency of Axiom A without cycles. As Palis [**Pa1**] showed, Ω-stability and Axiom A together already imply the absence of cycles. After Mañé's proof of the stability theorem, his arguments were found by Palis to yield the Ω-stability theorem as well [**Pa2**].

d. Stability of flows. For flows there are analogous sufficient conditions for Ω-stability [**PS1**] and structural stability [**Ro2**]. But the analogous stability and Ω-stability conjectures resisted for another decade beyond Mañé's work. Both were settled by Hayashi in 1997 [**Hy**], who developed the Connecting Lemma (Theorem 3.2.1) for the purpose. This is a major extension of the Pugh Closing Lemma [**S-HK**] and also simplifies the proof of Mañé's result for diffeomorphisms. An alternate proof of the stability conjecture for flows was given by Wen [**Wen**], using the Connecting Lemma as well. Earlier, Hu [**Hu**] had proved this result in dimension three. The central problem in passing from discrete to continuous time is the possibility of fixed points on which periodic points accumulate. In such a situation the arguments for the discrete-time case cannot be applied. Hayashi's Connecting Lemma can be used to show that this is an unstable situation, hence ruled out by the assumption of structural stability.

e. Classification on infranilmanifolds. All known basic examples of Anosov diffeomorphisms are automorphisms of a torus or of an infranilmanifold (Subsection 2.1d). By structural stability, C^1 perturbations give further examples. On these manifolds there are no other examples by the following classification theorem:

THEOREM 3.5.3 ([**Fr, Mn2**]). *Every Anosov diffeomorphism of an infranilmanifold is topologically conjugate to a hyperbolic automorphism.*

This classifies all Anosov diffeomorphisms hitherto known. In particular, it means that no topologically new Anosov diffeomorphisms can be found on infranilmanifolds. An interesting example in this regard is that of a (codimension one) Anosov diffeomorphism on a manifold that is topologically a torus but not diffeomorphic to one [**FJ**]. It is defined on $\Sigma^n \# \mathbb{T}^n$, where Σ^n is a nonstandard homotopy n-sphere and $n > 4$.

Franks [**Fr**] proved this result on tori and assuming regional recurrence [**S-HK**] (which is definitely necessary). Manning [**Mn2**] generalized the arguments to the infranilmanifold setting and showed that transitivity is automatic. An outline of the proof goes as follows: Any Anosov diffeomorphism is homotopic to a linear map (given by the action on homology). Show that this map is hyperbolic. A precise exponential asymptotic of periodic orbit growth resulting from the spectral decomposition (see Subsection 3.3a) and the Lefschetz Fixed-Point Formula play the key roles. Having noted that within the homotopy class of a hyperbolic automorphism any map has the linear model as a factor (via a map respecting the stable/unstable foliations) one finishes by showing that the semiconjugacy to the linear model is indeed injective, hence a homeomorphism. See [**KH**] for a presentation in the toral case. An intermediate step of independent interest is Manning's proof of regional recurrence, which implies mixing by the spectral decomposition. Note also Theorem 4.2.2 below, where regularity hypotheses imply the infranilmanifold structure.

f. Manifolds that admit Anosov systems. The classification raises several obvious but still open questions. First of all, which (infra)nilmanifolds admit Anosov diffeomorphisms? It turns out to be precisely those whose fundamental group admits an automorphism without one as an eigenvalue. In dimension six these have been classified, in dimension five or less there are only tori [**It**]. A special class is described by [**DM**] (infranilmanifolds of nilpotency class two and in whose fundamental group the commutator subgroup has maximal torsion-free rank admit an Anosov diffeomorphism if and only if the torsion-free rank of the abelianization of the fundamental group is at least three). Second, are there any other manifolds that admit Anosov diffeomorphisms? Flat manifolds of this kind have been characterized algebraically by Porteous [**Pr**] in terms of a standard representation of the linear holonomy group in $GL(n,\mathbb{Z})$. Necessary conditions have been obtained by using that the existence of stable and unstable foliations imposes topological restrictions. Hirsch [**Hi**] obtained such results, as did Shiraiwa [**Shi**], who gives a list of manifolds that do not admit Anosov diffeomorphisms. A result by Yano [**Y**] is that there are no regionally recurrent Anosov diffeomorphisms on negatively curved manifolds.

An interesting observation is due to Brin [**Br2**]: For automorphisms of infranilmanifolds there are two conditions on the Mather spectrum, each of which imply that the manifold is a torus.

g. Codimension 1 Anosov diffeomorphisms. In the aforementioned paper Franks [**Fr**] also showed that regionally recurrent codimension one Anosov diffeomorphisms (*i.e.*, those with one-dimensional stable or unstable manifolds) are topologically conjugate to a linear toral automorphism; Newhouse [**Nh**] showed that regional recurrence follows from the codimension one assumption. Thus

THEOREM 3.5.4. *Codimension one Anosov diffeomorphisms are conjugate to linear toral automorphisms.*

h. Holomorphic Anosov diffeomorphisms. For holomorphic Anosov diffeomorphisms some of the classification goes through. According to a result by Ghys [**Gh4**], if f is a transitive complex-codimension one holomorphic Anosov diffeomorphism of a compact complex manifold M then M is homeomorphic to a torus and f is topologically conjugate to a linear automorphism. In this case a new facet is that one may have holomorphic (rather than topological) rigidity. The same paper shows that if f is a holomorphic Anosov diffeomorphism of a compact complex surface M then M is a complex torus and f is holomorphically conjugate to a linear automorphism. Note that in particular the invariant foliations are holomorphic.

As a special case of the situation of complex-codimension one flows Ghys classifies holomorphic Anosov flows on compact complex 3-dimensional manifolds up to finite covers.

i. Codimension one flows. An analogous classification for Anosov flows is entirely lacking. One basic difference is that Anosov flows, unlike any known Anosov diffeomorphisms, may fail to be regionally recurrent, as shown by an example of Franks and Williams [**FrW**]. This has reduced hopes for a classification to those classes of Anosov flows for which regional recurrence can be guaranteed. Among these are codimension one Anosov flows on manifolds of dimension at least four [**Vj1**].

The *Verjovsky–Ghys Conjecture* [**Vj1, Gh2**] states that codimension one flows in dimension greater than 3 have a global cross-section, *i.e.*, are special flows over an Anosov diffeomorphism, which by Theorem 3.5.4 is topologically conjugate to a toral automorphism. (Verjovsky conjectured this for free polycyclic fundamental group.) Thus codimension one Anosov flows would be classified up to orbit equivalence.

Although there has been progress towards this conjecture, it remains open. It has been proved under additional assumptions such as solvable fundamental group [**Pl2**], fundamental group that admits a normal abelian noncyclic subgroup [**Bb1**], or volume preservation plus either differentiability (or almost Lipschitzness) of the strong stable and unstable foliations (in which case one gets constant return time) or almost C^2 codimension one (weak) foliation [**Gh2, Si**].

There is quite a bit of structural theory [**Vj2**]. For example, the universal cover of a manifold with a codimension one Anosov flow is a Euclidean space, and the lift of the flow is conjugate to the flow generated by $\partial/\partial x_1$ [**Pm**]. Indeed, M is an Eilenberg–MacLane space $K(\pi_1(M), 1)$, $\pi_1(M)$ has finite cohomology dimension, is torsion free and has exponential growth. If $\pi_1(M)$ has nontrivial center then $\dim M = 3$ and M is orientable, with the flow topologically conjugate, up to finite cover, to the geodesic flow of a compact hyperbolic surface [**Vj1**]. An Anosov flow on a 3-manifold is, up to finite cover, conjugate to a geodesic flow on a surface if the stable foliation is C^2 [**Gh1**] or the holonomies are restrictions of projective maps of the circle [**Bb2**].

Ghys [**Gh3**] classifies volume-preserving Anosov flows on 3-manifolds with smooth invariant foliations into suspensions of hyperbolic automorphisms of the torus and geodesic flows on surfaces of constant negative curvature (up to finite coverings) as well as a new type of flow that differs from the old ones by a special time change. This was the starting point for smooth rigidity theory (Subsection 4.2c).

j. Geodesic flows. For brevity a Riemannian metric with Anosov geodesic flow is said to be an Anosov metric. A manifold with an Anosov metric is said to be an Anosov manifold. In order to pursue anything like a classification of geodesic Anosov flows one needs to characterize when a geodesic flow is Anosov. In particular one should establish how Anosov metrics relate to those of negative curvature and which smooth manifolds admit Anosov metrics. Klingenberg addressed this last question, proving that for an Anosov manifold M there are no conjugate points, every closed geodesic has index 0, the universal cover is a disk, $\pi_1(M)$ has exponential growth, every nontrivial abelian subgroup of $\pi_1(M)$ is infinite cyclic, the geodesic flow is ergodic and periodic orbits are dense [**Kl1**]. The deep fact that there are no conjugate points was proved under weaker assumptions (existence of an invariant Lagrangian subbundle) by Mañé [**M2**] using the Maslov index, and the remaining properties follow. An elementary proof, for surfaces, of this stronger fact is in [**Pt3**, Section 2.6], using only that there is no global cross-section. (The latter fact, together with other important ones about geodesic flows, rests on exactness of the symplectic form.) The properties just listed are the main properties known for geodesic flows of negatively curved manifolds, so this result suggests the *uniformization conjecture*: Anosov manifolds admit a negatively curved metric. (It should be noted that Anosov metrics need not be negatively curved [**Ho2**].)

Eberlein [**E**] has done much work to characterize Anosov metrics. For example: For compact (M, g) without conjugate points he describes stable and unstable spaces of perpendicular Jacobi fields and shows equivalence of complementarity of these, the metric being Anosov, and vanishing of all bounded perpendicular Jacobi fields. If g has no focal points, *i.e.*, the length of a perpendicular Jacobi field with initial value zero is increasing, then uniform exponential growth of all such perpendicular Jacobi fields implies that g is Anosov.

We refer to [**S-K**] for a better account of this issue.

6. Invariant measures

Much of what might be surveyed here is contained in the Survey by Chernov [**S-C**], some other subjects have been described in [**S-HK**]. Therefore we now mainly point to the appropriate references.

a. Periodic measures. (See also [**KH**].) For hyperbolic sets (or expansive systems with specification) an abundance of ergodic measures arises from measures concentrated on periodic orbits. Many important invariant measures are constructed by approximation with such measures (Subsection 3.6c).

b. Markov partitions. The definition, construction and significance of Markov partitions is amply described in [**S-C**]. As was also previewed in [**S-HK**], this device establishes that for many purposes hyperbolic dynamics is modeled by a symbolic system. In the topological category this gives precise estimates on the growth of the number of periodic points, for example. For measure-theoretic aspects this makes earlier results about invariant measures for symbolic systems immediately applicable. This makes the use of Markov partitions effective in many ways and many facts in hyperbolic dynamics are relatively easy consequences.

c. Equilibrium states. Equilibrium states were defined in [**S-HK**] as measures maximizing pressure. Entropy is a special case and the measure of maximal entropy of an Anosov system is known as the Bowen–Margulis measure (see Subsection 3.6d). In [**S-C**] equilibrium states are obtained using Markov partitions. On the other hand, as outlined in [**S-HK**], one can prove existence and uniqueness (as well as mixing) of equilibrium states using expansivity and specification. This was first done by Bowen, and the elegance of this argument lies not least in the fact that it is monolithic, as opposed to dividing the work between a reduction to the symbolic case and dealing with the latter. His arguments are rendered fully in [**KH**]. Recently Knieper [**S-K, Kp**] has shown that there is a unique measure of maximal entropy also for the case of geodesic flow on rank-1 manifolds. These are manifolds of nonpositive curvature where no geodesic admits a parallel Jacobi field other than its tangent field. This is the dynamically natural generalization of negative curvature and is coherent with the eponymous notion for locally symmetric spaces.

Equilibrium states are not only mixing, but define K-systems [**Ru1, Sn**] and are Bernoulli [**G, L1, R, Bu**].

d. The Margulis measure. The Margulis measure is the unique measure of maximal entropy for an Anosov flow or diffeomorphism, but the construction given by Margulis [**Mg, KH**] is completely different. We give a crude outline here.

For an Anosov flow φ^t on a manifold M consider the space $C(W^{0u})$ of functions on M with compact support in a single weak-unstable leaf that restrict to a continuous function on that leaf. While this is not a linear space, one can scale functions and add any two functions whose supports are on the same leaf. Fix a function f_0 among these whose support has nonempty interior in the corresponding leaf. For $t \geq 0$ define the functional $F_t(f) := \int f \circ \varphi^{-t} \, d\lambda^{0u}$, where λ^u is the Riemannian volume on the appropriate weak-unstable leaf. Now set $C^* = \{F = \sum_{i=1}^m c_i F_{t_i} \mid i \in \mathbb{N}, c_i, t_i \geq 0, F(f_0) = 1\}$ and $\widehat{\varphi^t}^*(F)(f) := F(f \circ \varphi^{-t})/F(f_0 \circ \varphi^{-t})$, the projectivization of $\varphi^{t^*}(F)(f) := F(f \circ \varphi^{-t})$. The Tychonoff Fixed Point Theorem gives an $m \in \overline{C^*}$ and $h > 0$ such that $\varphi^{t^*}(m) = h^t m$. One can view m as the asymptotic normalized pullback of Lebesgue measure on unstable leaves. It indeed defines leafwise measures μ^{0u}, and these are holonomy-equivariant.

An analogous construction for strong stable leaves gives corresponding measures μ^s, although these are only approximately holonomy-equivariant. (For sufficiently nearby leaves the values on holonomy-related functions have ratio close to 1.)

From these measures on leaves we now construct a finite φ^t-invariant measure on M by locally defining a *weighted product measure*. Every $p \in M$ has a neighborhood $U(p)$ which is a *local product cube*, *i.e.*, using the local product structure we can write $U(p)$ as $U^{0u}(p) \times U^s(p)$, where $U^{0u}(p) \subset W^{0u}(p)$ and $U^s(p) \subset W^s(p)$. If $O \subset U(p)$ let

$$f_O(q) := \mu^s((\{q\} \times U^s(p)) \cap O) \quad (q \in U^{0u}(p)).$$

For $q \in U^s(p)$, $A \subset U^{0u}(p)$ let $\mu_q(A) := \mu^{0u}(A \times \{q\})$ wherever defined. This is independent of $q \in U^s(p)$, so $\mu(O) := \int f_O(x) \, d\mu_q(x)$ is well defined. The measure on M obtained by extending to Borel sets is finite and φ^t-invariant. Normalizing it, we obtain the Margulis measure.

The Margulis measure is characterized by the uniform exponential scaling of its conditionals under the flow. The scaling constant h is the topological entropy. This directly leads to estimates of the Margulis measure of Bowen balls that show absolute continuity with respect to the Bowen measure, and hence equality by ergodicity [**T**], [**KH**, Theorem 20.5.15]. Furthermore, the holonomy-equivariance also characterizes the Margulis measure [**BM**]. Finally, Hamenstädt gave a new description of this measure by showing that its unstable conditionals are spherical measures for a distance defined through dynamical information [**Hs1**].

e. The Sinai–Ruelle–Bowen measure. A particularly interesting equilibrium state is the measure of maximal pressure for the logarithm of the unstable Jacobian. It is known as the Sinai–Ruelle–Bowen measure or SRB-measure. Suppose f is a diffeomorphism of a compact manifold M and Λ a transitive Axiom A attractor with basin U, for example $\Lambda = U = M$. (This means Λ is a compact locally maximal hyperbolic attractor and periodic points of $f\restriction_\Lambda$ are dense in Λ.) Then the following are equivalent [**Sn, Ru2, BR, Bw1, Yo**]:

1. μ is an equilibrium state for the logarithm of the unstable Jacobian,
2. The conditionals of μ induced on unstable leaves are absolutely continuous with respect to Riemannian volume on each leaf,
3. For Lebesgue-a.e. $x \in U$ and for every $\varphi \in C^0(M)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \int \varphi \, d\mu,$$

i.e., Lebesgue-a.e. point is μ-typical.

For a further characterization see Section 5.9.

(3) follows from (2) because of absolute continuity of the stable foliation (this is also true in the nonuniformly hyperbolic situation so long as there are no zero exponents and the measure is ergodic). To see this, picture a piece of the attractor as a local unstable leaf. Then μ-a.e. point is μ-typical. Since μ has absolutely continuous conditionals, this means that Lebesgue-a.e. point on the attractor is μ-typical. Then for every such point its entire stable leaf consists of points that are μ-typical, *i.e.*, Lebesgue-a.e. stable leaf is μ-typical, and by absolutely continuity of the foliations, Lebesgue-a.e. point in the basin is μ-typical.

The first of these three characterizations of the Sinai–Ruelle–Bowen measure is interesting because for connected attractors it gives existence and strong stochastic properties (mixing, K, Bernoulli) right away (Subsection 3.6c, [**Ru1, Sn, G, L1, R, Bu**]). Note that for an expanding map this simply means absolute continuity of the invariant measure. Indeed, any smooth invariant measure for a hyperbolic dynamical system is a Sinai–Ruelle–Bowen measure. By uniqueness of this measure the second characterization shows that only one invariant Borel probability measure has absolutely continuous conditionals. The last characterization has met with the widest interest, however.

It is interpreted as evidence that computed pictures of such an attractor are accurate because almost every sufficiently long orbit segment gives a correct picture of the attractor, the density variations reflecting those of this preferred measure. One has to be a bit careful for two reasons: A computer might pick up only exceptional orbits, such as in the case of the map $x \mapsto 2x \pmod 1$, where a binary computer will always render the atom at zero rather than anything like Lebesgue measure, which is the SRB-measure for this map (Subsection 3.2c). The other problem is that computers produce pseudo-orbits, about whose distribution nothing has been said. One needs a statement about the effects on the SRB-measure of superimposing noise on actual orbits. For some kinds of noise there are results to the effect that the asymptotic distribution of almost all noisy orbits will approximate the SRB-measure. Thus, for objects proven to be Axiom A attractors it is usually taken for granted that computed pictures are meaningful. Accordingly, the Sinai–Ruelle–Bowen measure has been called the “physically observed” measure.

In the case of transitive Anosov diffeomorphisms this measure is also of interest even though the manifold is an attractor in the trivial sense. In this case there is a “complementary” measure obtained by considering the stable Jacobian. These two coincide if and only if one of them is absolutely continuous (Subsection 4.1a has a criterion for when this is the case). Therefore, Anosov systems are either volume-preserving or completely dissipative, *i.e.*, successive pullbacks of volume converge to a singular measure, the Sinai–Ruelle–Bowen measure. The two-dimensional situation is of particularly outstanding interest because in this case the Sinai–Ruelle–Bowen measure provides preferred parametrizations of the (one-dimensional) unstable leaves.

7. Partial hyperbolicity

Partial hyperbolicity requires $\lambda < 1$ *or* $\mu > 1$ in the Stable Manifold Theorem (Subsection 2.2a), but not both. Indeed, one requires $TM = E^+ \oplus E^0 \oplus E^-$ such that $\|Df^{-n}\restriction_{E^+}\|\|Df^n\restriction_{E^0}\| \le \lambda < 1$ and $\|Df^{-n}\restriction_{E^0}\|\|Df^n\restriction_{E^-}\| \le \lambda$ for $n \in \mathbb{N}$, *i.e.*, E^+ expands more than anything in E^0 and E^- contracts more than anything in E^0. Furthermore, one usually requires that $Df^n\restriction_{E^0}$ has subexponential growth. In this setting there are nontrivial center manifolds (unless $E^0 = \{0\}$). The extent to which these systems can be understood is limited by the fact that no restriction is imposed on the "subexponential part" of their behavior. For example, the product of any dynamical system with only subexponential expansion with a hyperbolic dynamical system is partially hyperbolic. Accordingly, hyperbolic methods may give some global insights but do not help study the nonhyperbolic factor. Partial hyperbolicity is almost always supplemented by assumptions that rule out products of a hyperbolic system with one that has no exponential behavior. Then hyperbolic techniques may resolve global issues that are dominated by the hyperbolic behavior. Stable ergodicity (Subsection 3.7c) is of this kind. As for nonuniformly hyperbolic systems, we can ask how much of the uniformly hyperbolic theory works in the partially hyperbolic situation.

This situation poses different challenges and has a very different flavor from the theory of hyperbolic dynamical systems, whether uniformly hyperbolic or not. Therefore this subject is surveyed separately [**S-B**]. In this section we only give a fleeting impression of the issues that arise and the concepts used to address them.

a. Structural results. As suggested earlier, there is little reason to expect much of the structural theory of the uniformly or nonuniformly hyperbolic situation to hold for all partially hyperbolic systems because the effects of the subexponential component in a partially hyperbolic system can be substantial.

Of expansivity, for example, there remains sensitive dependence on initial conditions, *i.e.*, for any point there are nearby points whose orbit moves away (simply make sure to arrange for nontrivial distance in the hyperbolic direction). Likewise, product examples show that closing and shadowing cannot be expected. If the subexponential direction is integrable then one might hope for orbits that at least return to the same subexponential leaf, even if not close to the starting point. Such results were obtained for geodesic flows on nonpositively curved manifolds [**BBES**].

There is a specific assumption on the invariant manifolds, accessibility, that carries much of the hyperbolic theory to this setting.

b. Invariant foliations and accessibility. In the partially hyperbolic situation the distributions E^+ and E^- are uniquely integrable to invariant laminations W^u and W^s, which satisfy

1. $T_xW^s(x) = E^-_x, \quad T_xW^u(x) = E^+_x;$
2. $f(W^s(x)) \subset W^s(f(x))$, $f^{-1}(W^u(x)) \subset W^u(f^{-1}(x))$;
3. for every $\delta > 0$ there exists $C(\delta)$ such that for $n \in \mathbb{N}$

$$d(f^n(x), f^n(y)) < C(\delta)(\lambda+\delta)^n d(x,y) \text{ for } y \in W^s(x),$$
$$d(f^{-n}(x), f^{-n}(y)) < C(\delta)(\mu-\delta)^{-n} d(x,y) \text{ for } y \in W^u(x).$$

The main difference to the hyperbolic case is that the dimensions of these leaves do not sum to that of the ambient manifold.

How to overcome this defect is best explained in the case of a dynamical system that is partially hyperbolic on a compact manifold M. In this case the invariant laminations are foliations. The model situation that illustrates how hyperbolic effects may dominate the dynamics is that in which the distributions $E^\pm$ are smooth and $E := E^+ \oplus E^-$ is totally nonintegrable. This means that the closure of the space of vector fields tangent to E under the Lie bracket is TM. This happens in numerous homogeneous systems, such as time one maps of geodesic flows of compact locally symmetric spaces of rank 1 [**KH**, Section 17.7] or left translations of compact factors $\mathrm{GL}(n, \mathbb{R})/\Gamma$ by the one-parameter subgroup e^{tA} for A diagonal with distinct elements [**KSp**].

Such systems have many properties similar to hyperbolic systems: Topological transitivity, ergodicity and the Bernoulli property of the main invariant measures and exponential decay of correlations for smooth functions. However, they usually have no periodic points.

The smoothness assumption of this discussion is fragile under perturbation, but it is not essential. Without it, one can assume the *accessibility property*: Any two points in the phase space can be connected by a path of finitely many segments, each inside a stable or unstable leaf. Put differently, the neutral direction is only locally meaningful, and one can move anywhere by a path that never has a neutral component. This requires no differentiability and produces the same local effect of connecting any two nearby points by a path that is piecewise tangent to the hyperbolic subspace. This is the key assumption for proving persistence of topological transitivity [**BrP**].

c. Stable ergodicity. Ergodicity of volume or even ergodic components of positive measure can not be expected in full generality, because this fails for products or time one maps of suspensions. However, ruling out situations of this kind does give results of some interest. Volume-preserving Anosov systems are *stably ergodic*, *i.e.,* all volume-preserving C^2 perturbations are ergodic. This observation has led to the question of which volume-preserving C^2 diffeomorphisms have this property. Partially hyperbolic systems that do not have an obvious product-like structure seem like a good candidate and have been studied in this regard, beginning with time one maps for geodesic flows of negatively curved manifolds [**W**].

Again, the required property is the accessibility property of the invariant foliations (Subsection 3.7b). So far it is known that volume-preserving partially hyperbolic systems are stably ergodic if they have the accessibility property and are dynamically coherent (the center distribution is integrable to a foliation whose leaves foliate the stable and unstable manifold of each of its elements). It is not known whether these additional hypotheses can be dropped, but experts conjecture that stable ergodicity is generic in the partially hyperbolic volume-preserving class [**GPS, PS4**]. In other words, volume is "prevalently" ergodic.

One conjectures furthermore that any open set of ergodic volume-preserving diffeomorphisms has an open dense subset of Bernoulli diffeomorphisms.

CHAPTER 4

Smooth conjugacy, moduli and rigidity

As pointed out in [**S-HK**], the natural conjugacy notion for smooth dynamical systems is topological conjugacy. This is particularly the case for hyperbolic dynamical systems because of the abundance of periodic points: If two dynamical systems are smoothly conjugate then their differentials at corresponding periodic points are conjugate linear maps and hence have the same eigenvalue data. Thus each periodic point carries a set of moduli (continuously varying invariants) of smooth conjugacy, together referred to as the Lyapunov cocycle. In particular, the topological rigidity (structural stability) of hyperbolic dynamical systems does not extend to smooth rigidity because the Lyapunov cocycle can be changed by C^∞ perturbations. Therefore it is especially interesting that there are situations in hyperbolic dynamics in which there is some smooth rigidity or in which smooth conjugacy classes can be described nicely by reasonable criteria.

To the extent that there are results about smooth conjugacy these mostly concern smooth conjugacy of a hyperbolic dynamical system with an algebraic model. Before continuing, therefore, it is well to note another obstruction to smooth conjugacy in this case. Algebraic models have transversely smooth subbundles and foliations and hence so does any smoothly conjugate system. This obstruction provides additional motivation for the works described in Section 2.3. It is of a different nature than the Lyapunov cocycle, but turns out not to be entirely independent of it. There are smooth rigidity results based on controlling only one of these aspects.

1. Cohomology, Lifschitz theory, regularity

To utilize the coincidence of the periodic data of two systems for proving smooth conjugacy one needs to pass to global information of a cohomological nature. Such cohomology is also important in other contexts such as triviality of time changes.

a. The Lifschitz Theorem. The main tool for obtaining global information from periodic data is the following result due to Lifschitz [**Ls**]:

THEOREM 4.1.1. *Let M be a smooth manifold, $f\colon U \to M$ an embedding of $U \subset M$, Λ a compact topologically transitive hyperbolic set, $\varphi\colon \Lambda \to \mathbb{R}$ Hölder continuous. If $\sum_{i=1}^{n} \varphi(f^i(x)) = 0$ whenever $f^n(x) = x$ then there is a function $\Phi\colon \Lambda \to \mathbb{R}$, unique up to an additive constant, such that $\varphi = \Phi \circ f - \Phi$. Moreover, Φ has the same Hölder exponent as φ.*

The Lifschitz Theorem gives expression to the abundance of periodic orbits provided by the Lefschetz fixed point formula and gives Lipschitz *transfer functions* when applied to Lipschitz functions; we call this the Lipschitz–Lifschitz Theorem.

The proof is easy: The transfer function Φ is determined along a dense orbit by choosing a value at one point of it. Uniform continuity follows from the Closing Lemma in Subsection 3.2b (using the Hölder hypothesis) as does Hölder regularity, so Φ extends to Λ.

For flows one gets an analogous result: If φ integrates to 0 over all periodic orbits then it is the derivative of some Φ in the flow direction. In other words, periodic data suffice to distinguish coboundaries among cocycles and thus provide global information.

From this basic result one can go in several directions. One is to consider cocycles φ with values in different groups, another is to ask how smoothness of φ gives smoothness of Φ. It is fairly clear that φ can be allowed to take values in abelian groups (with invariant metric, *e.g.,* compact). Indeed, for groups with a bi-invariant metric an analogous result holds for those φ taking values in a sufficiently small neighborhood of the identity. Finally, it also holds for simply connected solvable Lie groups.

Concerning the regularity of the cocycle and the transfer function there is a wide array of results. If $f \in C^2$ and $\varphi \in C^1$ then $\Phi \in C^1$. This slight strengthening of the Lipschitz–Lifschitz Theorem was proved by Lifschitz and Sinai [**LS**] in order to show that a transitive Anosov system preserves an absolutely continuous measure if and only if it has unit Jacobian over all periodic orbits. Lifschitz [**Ls**] showed that if f is a toral automorphism and $\varphi \in C^{r+\alpha}$ for some $r \in \mathbb{N}$, $\alpha \in (0,1)$ and there is an L^1 transfer function then there is a $C^{r+\alpha}$ transfer function. This was proved in [**LMM1**] in full generality, *i.e.,* if $\varphi \in C^\alpha$ with $\alpha \in \mathbb{R}^+ \smallsetminus \mathbb{N}_0$ or $\alpha \in \{1, \infty, \omega\}$ then $\Phi \in C^\alpha$. Here C^ω denotes analyticity. This partially extends even to nonuniformly hyperbolic systems [**dlLl**].

A detailed treatment of further extensions of Lifschitz theory is outside the purview of this survey. Various extensions were developed by Niţică and Török [**NT**], and S. Katok [**Ka**] obtained an approximate finitary version. A surprising result by Veech [**V**] has the same conclusion (for C^∞ functions) in a nonhyperbolic setting. One may naturally investigate analogous questions for actions of groups other than $\mathbb{Z}$ or $\mathbb{R}$, and there are results for the partially hyperbolic case (*e.g.,* [**KK**]).

b. Smooth conjugacy and periodic data. The lowest-dimensional situations are also those where periodic data alone give the strongest conclusions. The following result is in the spirit of the preceding discussion.

THEOREM 4.1.2. *Consider two Anosov diffeomorphisms of a compact surface or two Anosov flows of a compact 3-manifold that are C^k ($k = 2, 3, \dots, \infty, \omega$) and topologically conjugate via h. If the eigenvalues at corresponding periodic orbits are the same or if h and h^{-1} are absolutely continuous with respect to Lebesgue measure then h and h^{-1} are $C^{k-\epsilon}$.*

This is proved in [**dlLl**], but smooth conjuacy for two-dimensional systems with same periodic data was known before [**LMM2**], where a continuous-time analog (for eigenvalue data) is also proved. Lifschitz theory shows that the contraction and expansion rates agree also at nonperiodic points; this renders the conjugacy smooth along stable and unstable leaves, which by Theorem 2.3.9 implies smooth conjugacy. de la Llave's technique allows an extension even to the nonuniformly hyperbolic setting. Notice that in these results there is no mention of regularity of the invariant foliations; this obstruction happens to automatically disappear. This is complementary to results below where regularity of the invariant foliations alone gives smooth conjugacy to an algebraic model.

It should be noted that this result is false in higher dimension. This is shown by examples with novel moduli of smooth conjugacy other than eigenvalue data [**dlLl**]. Those moduli are related to Fourier coefficients on the torus.

c. Weak smooth equivalence and periodic data. We say that two diffeomorphisms $f^{(i)}\colon M^i \to M^i$, ($i = 1, 2$) are weakly C^r equivalent (or have the same germ extension) if

1. they are topologically conjugate via $h\colon M^1 \to M^2$
2. M^i is covered by C^r charts $\{(U^i_p, \varphi^i_p) \mid p \in M^i\}$ such that $p \in U^i_p$, $\varphi^i_p(p) = 0$ and φ^i_p depends continuously on p in the C^r topology ($i = 1, 2$)
3. $\varphi^1_{f^{(1)}(p)} \circ f^{(1)} \circ (\varphi^1_p)^{-1} = \varphi^2_{f^{(2)}(h(p))} \circ f^{(2)} \circ (\varphi^2_{h(p)})^{-1}$ near 0 for all $p \in M^1$.

In other words, $f^{(1)}$ and $f^{(2)}$ have the same coordinate representations at corresponding points. The nontrivial requirement is, of course, that these local charts depend continuously on the point. One can make an analogous definition for flows.

Clearly weak equivalence follows from smooth equivalence and implies coincidence of the Lyapunov cocycles. In particular, this equivalence relation shares a set of moduli with smooth conjugacy that was not yet described: The higher jet data at coresponding periodic points.

In several instances—smooth rigidity on low-dimensional tori [**LMM1, HK, FlK**] and of actions of $SL(n,\mathbb{Z})$ [**KL**]—the step from weak equivalence to smooth conjugacy was carried out successfully. Thus there are some natural settings where it is known that weak equivalence implies smooth conjugacy.

On the other hand, the examples of de la Llave that address problems with the Lyapunov cocycle [**dlLl**] also show that weak smooth equivalence does not imply smooth equivalence. It is an open question whether weak smooth equivalence might nevertheless provide an avenue towards new rigidity results.

2. Smooth rigidity and invariant structures

As noted above, smoothness of invariant structures associated with a hyperbolic dynamical system is necessary for smooth conjugacy to an algebraic model. There are several important instances where such conditions are sufficient.

Smoothness of the invariant foliations of a hyperbolic dynamical system has turned out to be sufficient for smooth conjugacy to an algebraic model in the symplectic case. For geodesic flows even more can be said. Open questions concern the precise amount of smoothness needed and possible conclusions in the absence of symplecticity.

a. Smoothness of the invariant foliations in dimension two. The most basic result in this direction is implicit: The proof by Avez [**Av**] that an area-preserving Anosov diffeomorphism of $\mathbb{T}^2$ is topologically conjugate to an automorphism actually gives a conjugacy as smooth as the invariant foliations. The definitive result in this setting is worth giving here, because it is suggestive of the work yet to be done in higher dimension.

THEOREM 4.2.1 ([**HK**]). *Let f be a C^∞ area-preserving Anosov diffeomorphism of $\mathbb{T}^2$. Then the invariant subbundles are differentiable and their first derivatives satisfy the Zygmund condition* [**Zy**, Section II.3, (3·1)] *and hence have modulus of continuity $O(x|\log x|)$* [**Zy**, Theorem (3·4)]. *There is a cocycle, the Anosov cocycle, which is a coboundary if and only if these derivatives have modulus of continuity $o(x|\log x|)$ or, equivalently, satisfy a "little Zygmund" condition. In this case, or if the derivatives have bounded variation* [**Gu**], *the invariant foliations are C^∞ and f is C^∞ conjugate to an automorphism.*

It is important to note how sharp the divide is between the general and the smoothly rigid situation. Indeed, the constant defining $O(x|\log x|)$ is nonzero a.e. except when the Anosov cocycle is trivial. Therefore this is the finest possible dichotomy.

To obtain C^∞ foliations it is actually shown first that triviality of the Anosov cocycle implies C^3 subbundles, and a separate argument then yields C^∞ foliations.

Following Guysinsky one can explain the Anosov cocycle using local normal forms. For a smooth area-preserving Anosov diffeomorphism on $\mathbb{T}^2$ deLatte [**dL**] showed that one can find local smooth coordinate systems around each point that depend continuously (actually C^1) on the point and bring the diffeomorphism f into the *Moser normal form* [**Mos1**]

$$f(x,y) = \begin{pmatrix} \lambda_p^{-1}x/\varphi_p(xy) \\ \lambda_p y\varphi_p(xy) \end{pmatrix},$$

where (x,y) are in local coordinates around a point p and the expression on the right is in coordinates around $f(p)$. The terms involving φ_p that depend on the product xy correspond to the natural resonance $\lambda_p\lambda_p^{-1} = 1$ that arises from area-preservation (actually from the family of resonances $\lambda_p = \lambda_p^{n+1}\lambda_p^{-n}$). The function φ_p is as smooth as f, and $\varphi_p(0) = 1$. Now we suppress the (continuous) dependence of λ and φ on p. Note that for a point $(0,y)$ we have

$$Df = \begin{pmatrix} \lambda^{-1}xy(1/\varphi)'(xy) + \lambda^{-1}/\varphi(xy) & \lambda^{-1}x^2(1/\varphi)'(xy) \\ \lambda y^2\varphi'(xy) & \lambda xy\varphi'(xy) + \lambda\varphi(xy) \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 \\ \lambda y^2\varphi'(0) & \lambda \end{pmatrix}.$$

In these local coordinates the unstable direction at a point $(0,y)$ on the stable leaf of p is spanned by a vector $(1,a(y))$. Since this subbundle is invariant under Df and since $f(0,y) = (0,\lambda y)$, the coordinate representation of Df from above gives $a(\lambda y) = \lambda^2y^2\varphi'(0) + \lambda^2a(y)$. If the unstable subbundle is C^2 then differentating this relation twice with respect to x at 0 gives $\lambda^2a''(0) = 2\lambda^2\varphi'(0) + \lambda^2a''(0)$, *i.e.*, $\varphi'(0) = 0$. This means that the Anosov obstruction is $\varphi'(0)$, where φ arises from the nonstationary Moser-deLatte normal form. (Thus this is also the obstruction to C^1 linearization.)

Hurder and Katok verify that $A(p) := \varphi_p'(0)$ is a cocycle and show that it is nonzero a.e. unless it is null-cohomologous. (Guysinsky's result that $C^{1+BV} \Rightarrow C^\infty$ follows because bounded variation implies differentiability almost everywhere.)

b. Smoothness in higher dimension.

1. *Nonconformality and obstructions.* The complications in higher dimension are due in large part to the simple fact that when the invariant foliations are not one-dimensional there may be different contraction and expansion rates at any given point. This effect is responsible already for the fact that in higher dimension the transverse regularity is usually lower than in the two-dimensional case, see Section 2.3. Note that the results there never assert higher regularity for both foliations than in the two-dimensional area-preserving case. If the obstruction vanishes that was used to show optimality of those results, then the regularity "jumps" up a little, and a further obstruction, associated with different contraction and expansion rates, may prohibit regularity $C^{1+O(x|\log x|)}$. Only when all those finitely many obstructions vanish can we have $C^{1+O(x|\log x|)}$. These obstructions are best described in normal form [**GuK**], as is the Anosov cocycle.

2. *Resonances.* To give a sample we show that a "1-2-resonance" produces an obstruction to C^1 foliations. To work with the simplest possible situation consider a 3-dimensional Anosov diffeomorphism f with fixed point p such that the eigenvalues $0 < \lambda < \mu < 1 < \eta < \infty$ of Df_p satisfy $\mu = \lambda\eta$. (This is a variant of the 1-2-resonance $\lambda_1 = \lambda_2^2$ for a symplectic system.) Up to higher-order terms that might arise from higher resonances the normal form of f at p is $f(x,y,z) = (\eta x, \mu y + axz, \lambda z)$. Representing E^u along the z-axis by $(1, v_1(z), v_2(z)$ gives

$$Df_{(0,0,z)}(1, v_1(z), v_2(z)) = (\eta, az + \mu v_1(z), \lambda v_2(z)),$$

which rescales to $(1, az/\eta + \mu v_1(z)/\eta, \lambda v_2(z)/\eta)$. Invariance of E^u therefore yields

$$v_1(\lambda z) = az/\eta + \mu v_1(z)/\eta.$$

Differentiating twice with respect to z gives $\lambda v_1'(0) = a/\eta + (\mu/\eta)v_1'(0)$, which implies $a = 0$ since $\lambda = \mu/\eta$. Thus the resonance term a in the normal form is an obstruction to C^1 Anosov splitting. (One can verify this without using normal forms, but the calculation is somewhat longer.) By the way, the work of Kanai mentioned below (Subsection 4.2c) made a rather stringent curvature pinching assumption to rule out a number of low resonances. The refinements by Feres and Katok that led to an almost complete proof of Theorem 4.2.7 centered on a careful study of the resonances that might arise without such pinching assumptions. This was delicate work because the issue are not only resonances at periodic points, but "almost resonances" between Lyapunov exponents. The papers [**FK, F1**] contain an impressive development of these ideas.

3. *The Anosov cocycle.* The next question is about an analog of the Anosov cocycle in higher dimension. While there is one, its vanishing is known to be necessary only for C^2 foliations [**Hb2**] and is not known to lead to higher regularity of the invariant foliations. Thus it has not yielded any effective application, and the central portion of the above approach falls apart.

4. *The bootstrap.* The bootstrap to C^∞ subbundles works in full generality, even without area-preservation, although it usually starts at regularity higher than C^3 (see Subsection 2.3f, [**Hb1, FL**]). In other words, once the invariant foliations have a sufficiently high degree of regularity, they are always C^∞.

c. Smooth rigidity. The main issue in higher dimension is to conclude from smoothness of the invariant foliations that there is a smooth conjugacy to an algebraic model, and to identify the right algebraic model in the first place. It is remarkable that the major rigidity results make no assumption that allows to use theorems such as the Franks–Manning classification (Theorem 3.5.3 and Theorem 3.5.4).

A result that appeared after systematic development of the continuous time situation (see also [**BFL**, Theorem 3]) will serve to illustrate this:

THEOREM 4.2.2 ([**BL**]). *Let M be a C^∞ manifold with an C^∞ affine connection ∇, $f\colon M \to M$ a topologically transitive Anosov diffeomorphism preserving ∇ with $E^u, E^s \in C^\infty$. Then f is C^∞ conjugate to an automorphism of an infranilmanifold. The invariant connection hypothesis can be replaced by invariance of a smooth symplectic form.*

Note the absence of a topological hypothesis. (There is a finite-smoothness sharpening of this result [**F2**] that does not use the powerful theorem of Gromov central to the proof by Benoist and Labourie.)

Now we turn to the continuous time case, where these developments are most significant.

1. *The ideas of Ghys and Kanai.* The history begins with the work of Ghys [**Gh3**], who classified volume-preserving Anosov flows on 3-manifolds with smooth invariant foliations into suspensions of hyperbolic automorphisms of the torus and geodesic flows on surfaces of constant negative curvature (up to finite coverings) as well as a new type of flow that differs from the old ones by a special time change. If the flow is known to be geodesic then the smooth conjugacy to the constant curvature geodesic flow preserves topological and metric entropies, and hence by entropy rigidity (Subsection 4.2d, [**K2**]) the original metric is constantly curved. The work towards classification of flows with smooth invariant foliations has followed this model closely. Before describing this, let us

mention in passing the secondary issue of reducing the regularity at which the classification becomes possible. In the situation of Ghys one can use an analysis of 3-dimensional volume-preserving Anosov flows and a result entirely analogous to Theorem 4.2.1 [**HK**] to conclude

THEOREM 4.2.3 ([**Gh3, HK**]). *A negatively curved metric on a compact surface is hyperbolic if its horocycle foliations are* $C^{1+o(x|\log x|)}$.

In higher dimension the seminal work is due to Kanai [**Kn**]. He was the first to implement the following strategy: If one assumes that the invariant foliations are smooth then one can study Lie bracket relations between the stable and unstable subbundles. The interaction between these and the dynamics can be used to build an invariant connection (named after him now [**S-K**]) and to show that it is flat, which in turn is used to build a Lie algebra structure that is identifiable as a standard model.

He obtained the following result:

THEOREM 4.2.4 ([**Kn**]). *The geodesic flow of a strictly 9/4-pinched negatively curved Riemannian metric on a compact manifold is smoothly conjugate to the geodesic flow of a hyperbolic manifold if the invariant foliations are* C^∞.

Two groups picked up this lead, with the primary aim of removing the pinching hypothesis, which in particular rules out nonconstantly curved locally symmetric spaces as models. It also emerged that the main import of the assumptions is dynamical rather than geometric, and that therefore one should look for theorems about flows more general than geodesic ones.

2. *Smooth rigidity*. Feres and Katok [**FK, F1**] built on Kanai's idea by refining his arguments with intricate analyses of resonance cases for Lyapunov exponents to cover most of the ground in terms of the admissible algebraic models.

THEOREM 4.2.5 ([**F1**]). *Consider a compact Riemannian manifold* M *of negative sectional curvature. Suppose the horospheric foliations are smooth. If the metric is 1/4-pinched or* M *has odd dimension then the geodesic flow is smoothly conjugate to that of a hyperbolic manifold. If the dimension is* $2 \pmod 4$ *then the geodesic flow is smoothly conjugate to that of a quotient of complex hyperbolic space.*

Some of the results proved along the way to this conclusion did not assume that the flow under consideration is geodesic. The refinements over Kanai's work were, in the case of the first hypothesis, a more delicate argument for vanishing of the curvature of the Kanai connection. Under the second hypothesis Feres shows that if the Kanai connection is not flat then the invariant subbundles split further (resonance considerations enter here), and a connection associated with this further splitting must be locally homogeneous.

Roughly simultaneously the complete result about smooth conjugacy was obtained by Benoist, Foulon and Labourie [**BFL**]. Not only does it include all geodesic flows, but it requires only a contact structure, which turned out to require substantial additional work. This makes it a proper counterpart of the three-dimensional result of Ghys:

THEOREM 4.2.6 ([**BFL**]). *Suppose* Φ *is a contact Anosov flow on a compact manifold of dimension greater than 3, with* C^∞ *Anosov splitting. Then there is an essentially unique time change and a finite cover on which the flow is* C^∞ *conjugate to the geodesic flow of a negatively curved manifold.*

What enables the authors to give a monolithic proof (as opposed to covering the various classes of symmetric spaces one by one) is a rigidity result by Gromov [**Gr, Bn, Z**].

This is the place where substantial regularity is needed, and on an m-dimensional manifold one can replace C^∞ in hypothesis and conclusion by C^k with $k \geq m^2 + m + 2$. This theorem is invoked in the first major step of the proof, to produce a homogeneous structure: The diffeomorphisms of the universal cover that respect the splitting and the flow form a Lie group that acts transitively. (Gromov's theorem produces this structure on an open dense set, and the Kanai connection is used to extend it.) Step two determines the structure of this group and its Lie algebra, and step three develops the dynamics of the group and relates it to the expected algebraic model.

The Feres–Katok approach needs a slightly different minimal regularity. In fact, if one adds the a posteriori redundant assumption of (nonstrict) 1/4-pinching (or merely strict 4/25-pinching) then C^5 horospheric foliations always force rigidity [**Hb1**].

We note an amplified version for the case of geodesic flows in which the conjugacy conclusion for geodesic flows is replaced by isometry of the metrics due to a more recent rigidity result by Besson, Courtois and Gallot, Theorem 4.2.10.

THEOREM 4.2.7. *If the horospheric foliations of a negatively curved compact Riemannian manifold are C^∞ then the metric is locally symmetric (up to isometry).*

It should be emphasized that the above result subsumes several classification steps. First of all, one obtains an orbit equivalence, which implies coincidence of the Lyapunov cocycles (periodic data). But furthermore, the original result in [**BFL**] directly arrives at a smooth conjugacy, which means that periods of periodic orbits are preserved as well. This is an extra collection of moduli for the continuous time case. Finally, in the case of geodesic flows, there is, in addition, the Besson–Courtois–Gallot Theorem 4.2.10, which gives the isometry.

3. *Related issues.* While the regularity of the invariant subbundles is usually substantially lower than in the two-dimensional case, it is widely believed that the minimal regularity for such smooth rigidity results should be C^2 or even $C^{1+\mathrm{Lip}}$, *i.e.,* quite close to that in Theorem 4.2.1. Indeed, these foliations are hardly ever $C^{1+\mathrm{Lip}}$ (Theorem 2.3.5), and if they are C^2 then the Liouville measure coincides with the Bowen–Margulis measure (Subsection 3.6c) of maximal entropy [**Hs3, S-K**]. (For Finsler metrics this is false [**Pt2**].) According to the Katok Entropy Rigidity Conjecture (Subsection 4.2d), this should imply that the manifold is locally symmetric. Optimists might suspect that rigidity already appears from $C^{1+o(x|\log x|)}$ or $C^{1+\text{“little Zygmund”}}$ on, but there is no evidence to that effect (save for Theorem 2.3.5).

Another result of Ursula Hamenstädt is worth remarking on here. It says that for contact Anosov flows with C^1 invariant foliations fixing the time parametrization fixes all other moduli of smooth conjugacy.

THEOREM 4.2.8 ([**Hs2**]). *If two conjugate (not just orbit equivalent) Anosov flows both have C^1 Anosov splitting and preserve a C^2 contact form then the conjugacy is C^2.*

The C^1 assumption on the splitting is not vacuous, but not stringent either, being satisfied by an open set of systems. Note that the conjugacy preserves both Lebesgue and Bowen–Margulis measure. If one keeps in mind that smooth conjugacy has been established mainly with one side being algebraic, this result is striking in its generality.

Inasmuch as they refer to flows, the hypotheses of the preceding rigidity results do not distinguish between the regularity of the strong versus weak invariant foliations. The reason is that for geodesic flows strong and weak foliations have the same regularity due to the invariant contact structure: The strong subbundles are obtained from the weak ones by intersecting with the kernel of the smooth canonical contact form.

Plante [**Pl1**] showed that the strong foliations may persistently fail to be C^1, namely when the asymptotic cycle of volume measure is nonzero. Even though the latter is not the case for (noncontact) perturbations of geodesic flows, these flows may still fail to have C^1 strong foliations (see [**Pt1, Bl**], where the contact form is "twisted" by an extra "magnetic force term", which does not produce a nontrivial asymptotic cycle).

d. Entropy rigidity. A different rigidity conjecture was put forward by Katok in a paper that proved it for surfaces [**K2**].

1. *The conjecture and its source.* The result that prompted the conjecture is that for the geodesic flow of a unit-area Riemannian metric without focal points on a surface of negative Euler characteristic E the Liouville and topological entropies lie on either side of $\sqrt{-2\pi E}$, with equality (on either side) only for constantly curved metrics. Katok [**K2**, p. 347] conjectured that Liouville measure has maximal entropy only for locally symmetric metrics, *i.e.,* that only in these cases do the topological and Liouville entropies agree. One can restate this as saying that equivalence of Bowen–Margulis and Lebesgue measure only occurs for locally symmetric spaces. This conjecture has engendered an enormous amount of activity and remains unresolved. The exact nature of the results in [**K2**] suggests some variants of this conjecture, however, that have been adressed more successfully.

2. *Results in special situations.* Flaminio [**Fl**] proved several interesting results in this regard. First of all, the conjecture holds locally along one-parameter perturbations of constantly curved metrics. On the other hand, in dimension 3 it is no longer the case that a hyperbolic metric (with unit volume) maximizes Liouville entropy. Therefore, the Katok entropy rigidity conjecture cannot take quite so neat a form as it does for surfaces. Foulon notes that for flows in dimension three it extends beyond the geodesic realm:

THEOREM 4.2.9 ([**Fo1**]). *A smooth contact Anosov flow on a three-manifold whose topological and Liouville entropies coincide is, up to finite covers, conjugate to the geodesic flow of a constantly curved compact surface.*

Foulon conjectures that three-dimensional C^∞ Anosov flows for which Bowen–Margulis and Lebesgue measure are equivalent must be C^∞ conjugate to either a suspension of a toral automorphism or the geodesic flow of a compact hyperbolic surface.

3. *General results.* That a metric is locally symmetric has been proved under a stronger but suggestive hypothesis [**L5**]. Consider the universal cover M of the manifold in question and for each $x \in M$ define a measure λ_x on the sphere at infinity by projecting the Lebesgue measure on the sphere $S_x M$ along geodesics starting at x (Lebesgue or visibility measure). Use a construction of the (Bowen–)Margulis measure [**Mg**] to define measures ν_x on the sphere at infinity [**Hs1**]. If there is a constant a such that $\lambda_x = a\nu_x$ for all x then M is symmetric (by [**L4, Yu**] it is asymptotically harmonic, by [**FL**] and Theorem 4.2.10 below it is symmetric).

In fact, one can also define a *harmonic* measure η_x at infinity for every $x \in M$ by using Brownian motion. In the case of surfaces its class coincides with the Lebesgue class only when the curvature is constant [**L3, K4**]. Musings by Sullivan [**Su**, p. 724] have led to the "Sullivan conjecture", analogous to the Katok conjecture, that in higher dimension the coincidence of the harmonic and visibility measure classes happens only for locally symmetric spaces.

If any two of these three measures here defined are proportional for every x then M is symmetric (again by [**L4, Yu, FL, BCG**]). The goal can be restated as the requirement to relax the hypothesis from proportionality to mutual absolute continuity [**L5**].

4. *The work of Besson–Courtois–Gallot.* Coming from rather a different direction, Besson, Courtois and Gallot found themselves addressing a related issue by showing that topological entropy is minimized only by locally symmetric metrics. Strictly speaking, their result concerns the volume growth entropy h of a compact Riemannian manifold, which is the exponential growth rate of the volume of a ball in the universal cover as a function of the radius. This is a lower bound for the topological entropy of the geodesic flow with equality if the sectional curvature is nonpositive [**Mn3**] (in fact, when there are no conjugate points [**FrM**]).

THEOREM 4.2.10 ([**BCG**]). *Let X, Y be compact oriented connected n-dimensional manifolds, $f\colon Y \to X$ continuous of nonzero degree. If g_0 is a negatively curved locally symmetric metric on X then every metric g on Y satisfies $h^n(Y,g)\operatorname{Vol}(Y,g) \geq |\deg(f)|h^n(X,g_0)\operatorname{Vol}(X,g_0)$ and for $n \geq 3$ equality occurs iff (Y,g) is locally symmetric (of the same type as (X,g_0)) and f is homotopic to a homothetic covering $(Y,g) \to (X,g_0)$. In particular, locally symmetric spaces minimize entropy when the volume is prescribed.*

5. *Magnetism.* A complementary result, about leaving the realm of geodesic flows, is contained in the work [**PP**] of the brothers Paternain: “Twisting” any Anosov geodesic flow (by adding a “magnetic” term to the Hamiltonian) strictly decreases topological entropy.

CHAPTER 5

The theory of nonuniformly hyperbolic systems

An introduction to the theory of nonuniform hyperbolicity can be found in [**S-BP, KH**], and [**S-K**] discusses some aspects related to geometry. A comprehensive treatment is in preparation [**BKP**]. This chapter aims to present the spirit of the work as well as some aspects of its present state. Key results and techniques are included. One should note right away that "*nonuniform*" here is used in the sense of "*no*t *n*ecessarily *uniform*", intending to generalize and include the theory of (uniformly) hyperbolic systems.

1. Contrast with the uniform case

Although this theory shares with that of uniformly hyperbolic systems the use of linearization and other aspects of smoothness of the dynamical system, one pervasive distinction is that the heart of the approach is in invariant measures. This may be viewed as the intrinsically natural generalization, but is also closely connected to the Oseledets Multiplicative Ergodic Theorem. Nevertheless, some results do not involve measure theory in their statements. A nice example is that a $C^{1+\alpha}$ diffeomorphism of a surface with positive topological entropy has a (hyperbolic) periodic point (Corollary 5.8.3, [**KH, P**]). This is an illustration of the fact that the theory of nonuniformly hyperbolic dynamical systems includes results that require no hyperbolicity assumption of any kind. Some of these hold in full generality (although they may be of limited usefulness when there is no hyperbolicity), others (such as the preceding sample) make mild assumptions that imply just enough hyperbolicity to yield nontrivial conclusions.

Before describing the theory of nonuniform hyperbolicity, it is good to recall the collection of facts that embody the hyperbolic paradigm in the uniform case: Expansivity, closing and shadowing lemma, Lifschitz theorem, spectral decomposition, Markov partitions, equilibrium states, absolute continuity of foliations, ergodicity of volume, the Bernoulli property of volume. This gives a list of desiderata for the theory of nonuniformly hyperbolic systems.

After introducing the framework in which the theory is developed, we give the structural results aimed at recovering the features just listed for the uniform case. This leads to a useful comparison between the two situations.

This chapter owes a great debt to the supplement by Katok and Mendoza in [**KH**]. Several results and proofs are directly adapted from there.

2. Lyapunov exponents

The definition of nonuniformly hyperbolic systems requires the notion of that of Lyapunov exponents. Even when one is not interested in the maximal possible generality, the natural setting for these is that of cocycles.

a. Cocycles. In this section let $f\colon X \to X$ be an invertible measure-preserving transformation of a Lebesgue probability space $(X, \mathcal{B}, \mu)$. (The Lebesgue assumption is not actually used frequently, but since it imposes only an extremely mild condition that is satisfied in all applications, we retain it throughout.) We call any measurable function $\mathcal{A}\colon X\times\mathbb{Z}\to GL(n,\mathbb{R})$ satisfying $\mathcal{A}(x,m+k)=\mathcal{A}(f^k(x),m)\mathcal{A}(x,k)$ a *measurable linear cocycle over f*, or simply a cocycle. Any cocycle $\mathcal{A}$ can be obtained from its *generator* by setting

$$\mathcal{A}(x,m)=\begin{cases}A(f^{m-1}(x))\cdots A(x) & \text{for } m>0,\\ A(f^m(x))^{-1}\cdots A(f^{-1}(x))^{-1} & \text{for } m<0,\\ \mathrm{Id} & \text{for } m=0.\end{cases}$$

Sometimes, if it does not cause confusion, we do not make a distinction between a cocycle and its generator and refer to the latter as a cocycle. A cocycle $\mathcal{A}$ over f induces a linear extension F of f to $X\times\mathbb{R}^n$ by $F(x,v):=(f(x),A(x)v)$.

Although not strictly necessary, it is illuminating to define an exponential splitting in a nonuniform way, working by analogy to the hyperbolic situation. To that end consider a sequence $(\langle\cdot,\cdot\rangle_m)_{m\in\mathbb{Z}}$ of inner products on $\mathbb{R}^n$ with associated norms $\|\cdot\|_m$ and angles $\measuredangle_m(\cdot,\cdot)$ between subspaces of $\mathbb{R}^n$.

DEFINITION 5.2.1. Given $\epsilon\ge 0<\lambda<\mu<\infty$ and a cocycle $\mathcal{A}$ over f, a point $x\in X$ is said to admit a (λ,μ,ϵ)-splitting if for each $m\in\mathbb{Z}$ there are positive numbers c_m and γ_m and a decomposition $E^s_m\oplus E^u_m=\mathbb{R}^n$ such that $\mathcal{A}(x,m)E^s_m=E^s_{m+1}$, $\mathcal{A}(x,m)E^u_m=E^u_{m+1}$ and for $k\in\mathbb{N}$

1. $\|\mathcal{A}(f^m(x),k)v\|_{m+k}\le c_{m+k}\lambda^k\|v\|_m$ for $v\in E^s_m$,
2. $\|\mathcal{A}(f^m(x),-k)v\|_{m-k}\le c_{m-k}\mu^{-k}\|v\|_m$ for $v\in E^s_m$,
3. $\measuredangle_m(E^s_m,E^u_m)\ge\gamma_m$,
4. $c_{m\pm k}\le c_m e^{\epsilon k}$ and $\gamma_{m\pm k}\ge\gamma_m e^{-\epsilon k}$.

Thus we require exponential estimates in subspaces that are transverse up to a subexponential degradation, and the degradation of the "contraction" and "expansion" estimates happens at a small exponential rate.

If $\lambda e^\epsilon<1$ we call E^s_m the stable subspace, and if $\mu e^\epsilon>1$ then E^u_m is called the unstable subspace. If both hold then x is said to be a hyperbolic orbit for $\mathcal{A}$.

The case of a (λ,μ) splitting from the (uniformly) hyperbolic situation is included here by taking $\epsilon=0$, which makes the exponential estimates and the transversality uniform. If $\mathcal{A}$ is the derivative cocycle and $\lambda<1<\mu$ are such that every x admits a (λ,μ)-splitting then X is a hyperbolic set.

b. Lyapunov exponents.

DEFINITION 5.2.2. For a cocycle $A\colon X\to GL(n,\mathbb{R})$ over a transformation $f\colon X\to X$ and for $(x,v)\in X\times\mathbb{R}^n$ the (possibly infinite) number

$$\bar\chi^+(x,v,\mathcal{A}):=\bar\chi^+(x,v):=\overline{\lim}_{m\to\infty}\frac{1}{m}\log\|\mathcal{A}(x,m)v\|$$

is called the *upper Lyapunov exponent of (x,v) with respect to the cocycle $\mathcal{A}$*. If $\lim_{m\to\infty}(1/m)\log\|\mathcal{A}(x,m)v\|$ exists then we denote it by $\chi^+(x,v)$ and call it the *Lyapunov exponent of (x,v) with respect to the cocycle $\mathcal{A}$*.

One can check that for a linear cocycle $\mathcal{A}$ over f:

1. $\bar\chi^+(x,v)=\bar\chi^+(x,\lambda v)$ for $(x,v)\in X\times\mathbb{R}$ and $\lambda\in\mathbb{R}\smallsetminus\{0\}$,
2. $\bar\chi^+(x,v+w)\le\max\{\bar\chi^+(x,v),\bar\chi^+(x,w)\}$ for $v,w\in\mathbb{R}^n$,
3. $\bar\chi^+(x,v+w)=\max\{\bar\chi^+(x,v),\bar\chi^+(x,w)\}$ if $\bar\chi^+(x,v)\ne\bar\chi^+(x,w)$.

This implies that for a cocycle $\mathcal{A}$, each real number χ and each $x \in X$ the set $E_\chi(x) = \{v \in \mathbb{R}^n \mid \bar{\chi}^+(x,v) \le \chi\}$ is a linear subspace of $\mathbb{R}^n$ and if $\chi_1 \ge \chi_2$ then $E_{\chi_2}(x) \subset E_{\chi_1}(x)$. Furthermore, for each $x \in X$ there exists an integer $k(x) \le n$ and a collection of numbers and linear subspaces

$$\chi_1(x) < \chi_2(x) < \cdots < \chi_{k(x)}(x), \quad \{0\} \subset E_{\chi_1}(x) \subset E_{\chi_2}(x) \subset \cdots \subset E_{\chi_{k(x)}}(x) = \mathbb{R}^n$$

such that if $v \in E_{\chi_{i+1}}(x) \smallsetminus E_{\chi_i}(x)$ then $\chi^+(x,v) = \chi_{i+1}(x)$. These numbers are called the *upper Lyapunov exponents at x with respect to the cocycle $\mathcal{A}$*, and the collection of linear subspaces is the *filtration at x associated to the cocycle $\mathcal{A}$*. The *multiplicity of the exponent* $\chi_i(x)$ is defined as $l_i(x) = \dim E_{\chi_i}(x) - \dim E_{\chi_{i-1}}(x)$, and the *spectrum of $\mathcal{A}$ at x* is the collection of pairs

$$\operatorname{Sp}_x \mathcal{A} = \{(\chi_i(x), l_i(x)) \mid i = 1, \dots, k(x)\}.$$

c. The Oseledets Multiplicative Ergodic Theorem. The Lyapunov exponents of an orbit $\mathcal{O}(x)$ are the exponential growth rates of vectors under iteration of the differential. In other words, $\chi^+(x,v) := \lim_{m\to\infty}(1/m)\log\|Df^n{}_{|_x} v\|$. The Oseledets Multiplicative Ergodic Theorem shows that with respect to an f-invariant Borel probability measure this is well-defined almost everywhere. Furthermore, at a given x, the Lyapunov exponent attains at most $\dim M$ different values and there is a Lyapunov decomposition into subspaces corresponding to the various Lyapunov exponents, whose dimension defines the multiplicity of the corresponding exponent. In fact a stronger property of *regularity* holds almost everywhere which can be briefly described by saying that all deviations from the limiting behavior are subexponential. (This appears to trace back as far as Perron [**Pn3**], see [**BP**].)

THEOREM 5.2.3 (Osedelec Multiplicative Ergodic Theorem, [**S-BP, Os, Ra, Wt2**]). *If (X,μ) is a Lebesgue space, $f\colon X \to X$ a measure-preserving transformation and $A\colon X \to \mathbb{R}^n$ a measurable cocycle over X with $\log^+\|A^{\pm 1}(x)\| \in L^1(X,\mu)$ then there is a set $Y \subset X$ such that $\mu(X \smallsetminus Y) = 0$ and for each $x \in Y$:*

1. *There exists a decomposition $\mathbb{R}^n = \bigoplus_{i=1}^{k(x)} H_i(x)$ that is invariant under the linear extension of f determined by A. The Lyapunov exponents $\chi_1(x) < \cdots < \chi_{k(x)}(x)$ exist and are f-invariant and*

$$\lim_{m\to\pm\infty} \frac{1}{|m|} \log \frac{\|\mathcal{A}(x,m)v\|}{\|v\|} = \pm\chi_i(x) \tag{5.2.1}$$

uniformly in $v \in H_i(x) \smallsetminus \{0\}$.

2. *For $S \subset N := \{1,\dots,k(x)\}$ let $H_S(x) := \bigoplus_{i\in S} H_i(x)$. Then*

$$\lim_{m\to\infty} \frac{1}{m} \log\sin|\measuredangle(H_S(f^m(x)), H_{N\smallsetminus S}(f^m(x)))| = 0.$$

3. Tempering and Lyapunov metrics

An important related device is that of *tempering* and Lyapunov metrics, which introduces coordinate changes that bring the differential into a block form adapted to the Lyapunov decomposition and the Lyapunov exponents. The price is a distortion (of lengths and angles) that may grow exponentially, but at an arbitrarily slow rate. The main benefit of using these unbounded but tempered coordinate changes is that they reduce local questions of relative behavior of orbits near a reference orbit satisfying the conclusion of the Multiplicative Ergodic Theorem to the uniformly hyperbolic setting (with respect to mildly oscillating distorting coordinate systems). This in particular allows one to use the Stable

Manifold Theorem directly. Thus, the Lyapunov metric we encountered early on in the hyperbolic theory as a convenience is a device of significant importance in nonuniformly hyperbolic dynamics.

DEFINITION 5.3.1. A measurable function $C\colon X \to GL(n,\mathbb{R})$ is said to be *tempered with respect to* f, or simply *tempered*, if

$$\lim_{m\to\infty} \frac{1}{m} \log \|C^{\pm 1}(f^m(x))\| = 0$$

for almost every $x \in X$. If $A, B\colon X \to GL(n,\mathbb{R})$ are measurable maps defining cocycles $\mathcal{A}, \mathcal{B}$ over f, then $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent* if there exists a measurable tempered function $C\colon X \to GL(n,\mathbb{R})$ such that

$$A(x) = C^{-1}(f(x))B(x)C(x)$$

for almost every $x \in X$.

This is clearly an equivalence relation and if two cocyles $\mathcal{A}, \mathcal{B}$ are equivalent we write $\mathcal{A} \sim \mathcal{B}$. Also we say that a cocycle $\mathcal{A}$ over f is *tempered* if its generator A is tempered.

THEOREM 5.3.2 (Osedelets–Pesin ϵ-Reduction Theorem). *Suppose (X,μ) is a Lebesgue space, $f\colon X \to X$ a measure preserving transformation and $A\colon X \to GL(n,\mathbb{R})$ a measurable cocycle with $\log^+ \|A^{\pm 1}(x)\| \in L^1(X,\mu)$. Then there exists a measurable f-invariant function $k\colon X \to \mathbb{N}$, and numbers $\chi_1(x),\dots,\chi_{k(x)}(x) \in \mathbb{R}$, $l_1(x),\dots,l_{k(x)}(x) \in \mathbb{N}$ depending measurably on x such that $\sum l_i(x) = n$ and for every $\epsilon > 0$ there is a tempered map $C_\epsilon\colon X \to GL(n,\mathbb{R})$, called the* Lyapunov change of coordinates, *with the property that for almost all $x \in X$ the cocycle*

$$A_\epsilon(x) = C_\epsilon^{-1}(f(x))A(x)C_\epsilon(x)$$

has the Lyapunov block form

$$A_\epsilon(x) = \begin{pmatrix} A_\epsilon^1(x) & & & \\ & A_\epsilon^2(x) & & \\ & & \ddots & \\ & & & A_\epsilon^{k(x)}(x) \end{pmatrix},$$

where each $A_\epsilon^i(x)$ is an $l_i(x) \times l_i(x)$ matrix and

$$\|A_\epsilon^i(x)^{-1}\|^{-1},\ \|A_\epsilon^i(x)\| \in [e^{\chi_i(x)-\epsilon}, e^{\chi_i(x)+\epsilon}].$$

Furthermore $k(x)$ and $\chi_i(x)$ are as in Theorem 5.2.3 and for almost every $x \in X$ we can decompose $\mathbb{R}^n$ as $\bigoplus_{i=1}^{k(x)} H_i(x)$ such that $l_i(x) = \dim H_i(x)$ and $C_\epsilon(x)$ sends the standard decomposition $\bigoplus_{i=1}^{k(x)} \mathbb{R}^{l_i(x)}$ to $\bigoplus_{i=1}^{k(x)} H_i(x)$.

Before proving this result we define the Lyapunov metric. Let $Y \subset X$ be the set provided by Theorem 5.2.3. Thus if $x \in Y$ then $\mathbb{R}^n = \bigoplus H_i(x)$. Given $\epsilon > 0$ define a new scalar product on $H_i(x)$ by

$$\langle u, v\rangle'_{x,i} := \sum_{m\in\mathbb{Z}} \langle \mathcal{A}(x,m)u, \mathcal{A}(x,m)v\rangle e^{-2m\chi_i(x)} e^{-2\epsilon|m|},$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^n$ and $(e^{-2m\chi_i(x)}e^{-2\epsilon|m|})_{m\in\mathbb{Z}}$ is called the *Pesin tempering kernel*. This is well-defined because by (5.2.1) there is a constant $C_i(x,\epsilon)$

such that $\|\mathcal{A}(x,m)v\| \le C_i(x,\epsilon)e^{m\chi_i(x)}e^{\epsilon|m|/2}\|v\|$ and therefore

$$\langle u,v\rangle'_{x,i} \le C_i^2(x,\epsilon)\sum_{m\in\mathbb{Z}} e^{-|m|\epsilon}.$$

Since $\mathcal{A}(x,m+1) = \mathcal{A}(f(x),m)A(x)$ we have

$$\begin{aligned}\langle A(x)v,A(x)v\rangle'_{f(x),i} &= \sum_{m\in\mathbb{Z}} \|\mathcal{A}(f(x),m)A(x)v\|^2 e^{-2m\chi_i(f(x))}e^{-2\epsilon|m|} \\ &= \sum_{m\in\mathbb{Z}} \|\mathcal{A}(x,m+1)v\|^2 e^{-2m\chi_i(x)}e^{-2\epsilon|m|} = \sum_{k\in\mathbb{Z}} \|\mathcal{A}(x,k)v\|^2 e^{-2k\chi_i(x)}e^{-2\epsilon|k|}e^{\psi}\end{aligned}$$

with $2\chi_i(x) - 2\epsilon \le \psi := 2\chi_i(x) - 2\epsilon(|k-1| - |k|) \le 2\chi_i(x) + 2\epsilon$. Therefore

$$e^{-2\epsilon} \le \frac{\langle A(x)v,A(x)v\rangle'_{f(x),i}}{e^{\chi_i(x)}\langle v,v\rangle'_{x,i}} \le e^{2\epsilon}. \tag{5.3.1}$$

DEFINITION 5.3.3. The scalar product

$$\langle u,v\rangle'_x := \sum_{i=1}^{k(x)} \langle v_i,u_i\rangle'_{x,i},$$

on $\mathbb{R}^n$, where v_i is the projection of v to $H_i(x)$ with respect to $\bigoplus_{j=1}^{k(x)} H_j(x)$, is called the *Lyapunov scalar product*, and the norm $\|\cdot\|'_x$ induced by it the *Lyapunov norm* or *Lyapunov metric*.

PROOF OF THEOREM 5.3.2. Define a positive symmetric matrix $C_\epsilon(x)$ by $\langle\cdot,\cdot\rangle'_x = \langle C_\epsilon(x)\cdot, C_\epsilon(x)\cdot\rangle$ and set $A_\epsilon(x) := C_\epsilon^{-1}(f(x))A(x)C_\epsilon(x)$. Thus if $u,v \in H_i(x)$, then

$$\begin{aligned}\langle A(x)u,A(x)v\rangle &= \langle C_\epsilon^{-1}(f(x))A(x)u, C_\epsilon^{-1}(f(x))A(x)v\rangle'_{f(x),i} \\ &= \langle A_\epsilon(x)C_\epsilon^{-1}(x)u, A_\epsilon(x)C_\epsilon^{-1}(x)v\rangle'_{f(x),i},\end{aligned}$$

so applying (5.3.1) to $v = C_\epsilon^{-1}(x)u$ we obtain

$$e^{2(\chi_i(x)-\epsilon)} \le \frac{\|A_\epsilon(x)v\|^2}{\|v\|^2} \le e^{2(\chi_i(x)+\epsilon)}.$$

It remains to prove that $C_\epsilon(x)$ is tempered. Since the angles between the different subspaces have a subexponential lower bound by part 2 of Theorem 5.2.3, it is enough to consider block matrices. Note that $\log\|A_\epsilon^{\pm1}\|$ is bounded and hence Theorem 5.2.3 can be applied to A_ϵ. Set $X_N = \{x \in X \mid \|C_\epsilon^{\pm1}(x)\| < N\}$. For any $N \in \mathbb{N}$, by the Poincaré Recurrence Theorem there exists $Y \subset X_N$ such that $\mu(X_N \setminus Y) = 0$ and for each $y \in Y$ there is a sequence $m_k \to \infty$ such that $f^{m_k}(y) \in Y$ for all k. Then

$$\|\mathcal{A}_\epsilon(y,m_k)\| \le \|C_\epsilon^{-1}(f^{m_k}(y))\|\|\mathcal{A}(y,m_k)\|\|C_\epsilon^{-1}(y)\|$$

and therefore for almost every point $y \in Y$ the spectra of A_ϵ and A are the same. Thus this holds for almost every $x \in X = \bigcup_{N\in\mathbb{N}} X_N$, and for these points $A_\epsilon(x){:=}C_\epsilon^{-1}(f(x))A(x)C_\epsilon(x)$ implies

$$\overline{\lim}_{n\to\infty}\frac{1}{n}\log\|C_\epsilon(f^n(x))\| = 0.$$

□

LEMMA 5.3.4 (Tempering-Kernel Lemma). *Let* (X,μ) *be a Lebesgue space,* $f\colon X\to X$ *a measure-preserving transformation and* $K\colon X\to\mathbb{R}$ *a positive measurable function such that* $\lim_{m\to\infty}(1/m)(\log K(f^m(x))=0$. *Then for any* $\epsilon>0$ *there exists a measurable* $K_\epsilon\colon X\to\mathbb{R}$ *such that*

$$K_\epsilon(x)>K(x) \text{ and } e^{-\epsilon}\le\frac{K_\epsilon(x)}{K_\epsilon(f(x))}\le e^\epsilon.$$

Applied to $\|C_\epsilon(x)\|$ this gives the following relation between the Euclidean norm and the Lyapunov norm: $\|\cdot\|\le\|\cdot\|'_x\le K_\epsilon(x)\|\cdot\|$ and $e^{-\epsilon}\le K_\epsilon(f(x))/K_\epsilon(x)\le e^\epsilon$.

4. Regular neighborhoods

Now we can apply this theory of measurable cocycles to diffeomorphisms of compact smooth manifolds. We assume that $f\in\mathrm{Diff}^{1+\alpha}(M)$ for some $\alpha>0$. This assumption is pervasive in the theory of nonuniformly hyperbolic dynamical systems. The Osedelec–Pesin ϵ-Reduction Theorem 5.3.2 says that for a given $\epsilon>0$ and for almost every $x\in M$ there is a linear transformation $C_\epsilon(x)\colon T_xM\to\mathbb{R}^n$ such that

$$D_\epsilon(x):=C_\epsilon(f(x))\circ D_xf\circ C_\epsilon^{-1}(x) \tag{5.4.1}$$

has the Lyapunov block form as in Theorem 5.3.2 and C_ϵ is a tempered function.

a. Existence of regular neighborhoods. We denote by $B(0,r)$ the standard Euclidean r-ball in $\mathbb{R}^n$ centered at the origin.

For almost every $x\in M$ there is a neighborhood on which f acts much like the linear map $D_\epsilon(x)$ in a neighborhood of the origin:

THEOREM 5.4.1 ([**Pe2**]). *Let* $f\in\mathrm{Diff}^{1+\alpha}(M)$, $\alpha>0$, $\dim M=n$ *and suppose* f *preserves a Borel probability measure* μ. *Then there exists a set* $\Lambda\subset M$ *of full measure such that for any* $\epsilon>0$ *we can find a tempered function* $q\colon\Lambda\to(0,1]$ *with* $e^{-\epsilon}<q(x)/q(f(x))<e^\epsilon$ *and embeddings* $\Psi_x\colon B(0,q(x))\to M$ *such that*

1. $\Psi_x(0)=x$,
2. *if* $f_x:=\Psi_{f(x)}^{-1}\circ f\circ\Psi_x\colon B(0,q(x))\to\mathbb{R}^n$, *then* D_0f_x *has the Lyapunov block form,*
3. $d_{C^1}(f_x,D_0f_x)<\epsilon$ *on* $B(0,q(x))$,
4. *there are* $K>0$ *and* $A\colon\Lambda\to\mathbb{R}$ *measurable with* $e^{-\epsilon}<A(f(x))/A(x)<e^\epsilon$ *and* $K^{-1}d(\Psi_x(y),\Psi_x(z))\le\|y-z\|\le A(x)d(\Psi_x(y),\Psi_x(z))$ *for* $y,z\in B(0,q(x))$.

DEFINITION 5.4.2. The points $x\in\Lambda$ are called *completely regular points*. The set $N(x):=\Psi_x(B(0,q(x)))$ is called a *regular neighborhood* of $x\in\Lambda$.

It is here and in establishing Hölder continuity of the invariant subbundles that the $C^{1+\alpha}$ assumption is needed.

b. Hyperbolic points, admissible manifolds, and the graph transform. Let M be a compact Riemannian manifold, $\alpha>0$, $f\in\mathrm{Diff}^{1+\alpha}(M)$ and μ an invariant Borel probability measure on M. By Theorem 5.2.3 there is a Borel set $\Lambda\subset M$ of full measure such that the Lyapunov exponents $\chi_1(x)<\chi_2(x)<\cdots<\chi_{k(x)}(x)$ are defined at every $x\in\Lambda$ and $T_xM=H_1(x)\oplus\cdots\oplus H_{k(x)}(x)$, where $H_i(x)$ is a linear subspace of v for which $\lim_{n\to\infty}(1/n)\log\|D_xf^n(v)\|=\chi_i(x)$.

In the linear theory there is no special value of the exponents—a constant rescaling of a cocycle changes the Lyapunov exponents but not the Osedelec decomposition or other structures associated to the cocycle. But using the linear theory for nonlinear systems requires localization procedures to transfer information from the linearization to the dynamical system itself. At this point 0 acquires a particular role as a value for the Lyapunov exponents and we need to pay special attention to the sign of the exponents.

Let $s(x) := \max\{s \in \mathbb{N} \mid \chi_i(x) < 0$ for $1 \le i \le s\}$, $u(x) := \min\{u \in \mathbb{N} \mid \chi_i(x) > 0$ for $u \le i \le k(x)\}$,

$$E^s(x) = H_1(x) \oplus \cdots \oplus H_{s(x)}(x),$$
$$E^0(x) = H_{s(x)+1}(x) \oplus \cdots \oplus H_{u(x)-1}(x), \text{ and}$$
$$E^u(x) = H_{u(x)}(x) \oplus \cdots \oplus H_{k(x)}(x).$$

Then $T_xM = E^s(x) \oplus E^0(x) \oplus E^u(x)$. We call these subspaces *stable*, *neutral*, and *unstable*, respectively. If $E^0(x) = \{0\}$ then we say that x is a *hyperbolic point* for f, *i.e.*, all the Lyapunov exponents of x are different from zero. There always is a hyperbolic point for surface diffeomorphisms leaving invariant an ergodic measure of positive entropy. From now on we only consider hyperbolic points.

To simplify the exposition assume that $\dim M = 2$ and the stable and unstable subspaces are one-dimensional. By the Osedelec–Pesin Theorem 5.3.2 and localization we can consider families of maps on $\mathbb{R}^2$, and identify the stable subspace with the x-axis and the unstable one with the y-axis. Let $R_\delta = [-\delta, \delta] \times [-\delta, \delta]$. The stage is now set for applying the Hadamard Graph Transform Method [**Pe2, Ru3, PS3**] or the Perron–Irwin method [**FHY, BP**] to produce stable and unstable manifolds. But for many purposes it is sufficient to use admissible manifolds:

DEFINITION 5.4.3. A 1-dimensional submanifold $V \subset R_\delta$ is called an *admissible (s, γ, δ)-manifold near 0* if $V = \operatorname{graph} \varphi = \{(\varphi(v), v) \mid v \in [-\delta, \delta]\}$, where $\varphi\colon [-\delta, \delta] \to [-\delta, \delta]$ is a C^1 map such that $\varphi(0) \le \delta/4$ and $|D\varphi| \le \gamma$.

Similarly, a 1-dimensional submanifold $V \subset R_\delta$ is called an *admissible (u, γ, δ)-manifold near 0* if $V = \operatorname{graph} \varphi = \{(v, \varphi(v)) \mid v \in [-\delta, \delta]\}$, where $\varphi\colon [-\delta, \delta] \to [-\delta, \delta]$ is a C^1 map such that $\varphi(0) \le \delta/4$ and $|D\varphi| \le \gamma$.

If x is a hyperbolic point and $R(x, \delta) := \Psi_x(R_\delta)$ we say that $W \subset R(x, \delta)$ is an *admissible (s, γ, δ)-manifold near x* if $W = \Psi_x(V)$ with V an *admissible (s, γ, δ)-manifold near 0*. Similarly define admissible (u, γ, δ)-manifolds near x.

Admissible manifolds may not be invariant, but admissibility is, as one can show via the Hadamard graph transform method:

PROPOSITION 5.4.4. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 diffeomorphism such that $f(0) = 0$ and $f(u, v) = (Au + h_1(u, v), Bv + h_2(u, v))$ for $(u, v) \in \mathbb{R}^2$, where $|D_z h_i| < \epsilon$ for $z \in \mathbb{R}^2$ $(i = 1, 2)$, $|A| < e^{-\chi}$, $|B^{-1}| < e^{-\chi}$ and $\chi > 0$, $\epsilon \in (0, 1)$ such that $1/(1 - 2\epsilon) \le 1 + 4\epsilon < e^\chi < 1/\epsilon$. If $\gamma \in (0, \epsilon e^{-\chi}]$ and V is an admissible (u, γ, δ)-manifold near 0 then $f(V)$ is an admissible (u, γ, δ)-manifold near 0 and there exists $\lambda > 1$ such that $\|f(y) - f(z)\| > \lambda \|y - z\|$ for $y, z \in V$.*

Admissible manifolds allow us to use some sort of local product structure in regular neighborhoods.

LEMMA 5.4.5. *If $\gamma < 1$ then any admissible (s, γ, δ)-manifold near 0 intersects any admissible (u, γ, δ)-manifold near 0 at exactly one point and the intersection is transverse.*

PROOF. If $\varphi^s, \varphi^u : \mathbb{R} \to \mathbb{R}$ are C^1 maps such that $\operatorname{graph} \varphi^s$ and $\operatorname{graph} \varphi^u$ are admissible (u, γ, δ)- and (s, γ, δ)-manifolds, respectively, then $\varphi^u \circ \varphi^s : \mathbb{R} \to \mathbb{R}$ is a contraction because $|\varphi^u \circ \varphi^s(v) - \varphi^u \circ \varphi^s(w)| \le \gamma^2 |v - w|$. By the Banach Contraction Principle there is a unique fixed point v. But $(v, \varphi^s(v)) = (\varphi^u \circ \varphi^s(v), \varphi^s(v))$ is obviously the only point in $\operatorname{graph} \varphi^u \cap \operatorname{graph} \varphi^s$.

Transversality: $(\eta, \xi) \in T_{(v,\varphi^s(v))} \operatorname{graph} \varphi^s \cap T_{(v,\varphi^s(v))} \operatorname{graph} \varphi^u$ implies $|\xi| \le \gamma|\eta|$ and $|\eta| \le \gamma|\xi|$, hence $(\eta, \xi) = (0, 0)$. □

COROLLARY 5.4.6. *Let $x \in M$ be a hyperbolic point for $f \in \operatorname{Diff}^{1+\alpha}(M)$. Then any admissible (s, γ)-manifold near x intersects any admissible (u, γ)-manifold near x transversely and in exactly one point.*

5. Hyperbolic measures

Up to this point there were no hyperbolicity assumptions of any kind. In fact, one of the strengths of the theory of nonuniformly hyperbolic systems is that it can make interesting statements about dynamical systems without any such assumption. On the other hand, the theory has brought new insights into uniformly hyperbolic dynamics as well. In this section we study measures for which all Lyapunov exponents are nonzero a.e.

a. Pesin sets.

DEFINITION 5.5.1. We say that an f-invariant Borel probability measure μ for $f \in \operatorname{Diff}^{1+\alpha}(M)$, $\alpha > 0$, is an *f-hyperbolic measure* if $E^0 = \{0\}$ a.e., *i.e.*, the Lyapunov exponents are nonzero a.e.

For a completely regular point $x \in M$ (Definition 5.4.2) let $r(x)$ be the radius of the maximal ball contained in the regular neighborhood $N(x)$; we say that $r(x)$ is the *size* of $N(x)$.

For hyperbolic measures, one has exponential behavior a.e., which is a much weaker assumption than uniform hyperbolicity. To extend the theory of uniformly hyperbolic systems to this situation, one uses that for any given hyperbolicity estimate (with fixed constants) there is a (possibly empty) set, where this estimate holds, and that the union of these sets is the entire regular set. In other words, there are sets of arbitrarily large measure, called *Pesin sets*, on which one has uniform hyperbolic estimates:

THEOREM 5.5.2. *Let M be a compact surface, $\alpha > 0$, $f \in \operatorname{Diff}^{1+\alpha}(M)$ and μ an f-hyperbolic measure. Then for any $\delta > 0$ there exists a compact set Λ_δ and $\epsilon = \epsilon(\delta) > 0$ such that $\mu(\Lambda_\delta) > 1 - \delta$ and the positive Lyapunov exponent χ, the functions D_ϵ from* (5.4.1)*, C_ϵ from the ϵ-Reduction Theorem 5.3.2 for $\epsilon = \epsilon(\delta)$ applied to $A = Df$, q from Theorem 5.4.1, r from Definition 5.5.1 and the splitting $T_xM = E^s(x) \oplus E^u(x)$ are continuous on Λ_δ.*

PROOF. By the Luzin Theorem there is a set Λ_δ^1 such that $\mu(\Lambda_\delta^1) > 1 - \delta/2$, $\chi\restriction_{\Lambda_\delta^1}$ is continuous and $\chi_\delta := \inf\{\chi(x) \mid x \in \Lambda_\delta^1\} > 0$. Take $0 < \epsilon < \chi_\delta/100$. Again by the Luzin Theorem there is a set Λ_δ^2 with $\mu(\Lambda_\delta^2) > 1 - \delta/2$ on which D_ϵ, q, r, E^s and E^u are continuous. Take $\Lambda_\delta = \Lambda_\delta^1 \cap \Lambda_\delta^2$. □

DEFINITION 5.5.3. For $\delta > 0$ we call Λ_δ a *δ-Pesin set*, or simply a Pesin set.

One of the difficulties is that these Pesin sets are not usually invariant. Nevertheless, one obtains measurable invariant laminations. Often it is easier and sufficient to work with admissible manifolds instead.

6. Stable manifolds

The Hadamard–Perron Theorem applied in the nonuniform case also gives invariant laminations, but instead of uniformity in the size of leaves there is a measurable lower bound only. The same goes for the angle between stable and unstable leaves. In the uniform case the picture of stable leaves along an unstable one can be arranged (via local coordinates) as a horizontal line (unstable leaf) crossed by vertical ones. The nonuniform situation is best imagined as a horizontal line with a "fence" of vertical line segments, in the gaps of which there are somewhat crooked short line segments, between which there are much shorter line segments, some of them possibly quite close to horizontal, *et c.*

Their lack of regularity nonwithstanding, the invariant foliations do retain absolute continuity [**Pe2, PS3**]. Among the consequences is that the ergodic decomposition of a hyperbolic measure consists of sets of positive measure. In particular, there are at most countably many components.

7. Structural theory

Several of the central results of the uniform theory have counterparts in this setting. Among these are the Closing Lemma (which produces a hyperbolic periodic point), the Shadowing Lemma, the existence of Markov partitions (which here are approximate, or infinite [**KT**]), and the Lifschitz Theorem. There is also a spectral decomposition of a Pesin set for a hyperbolic measure into a finite union of orbit closures. While there is no structural stability, a vestige of it remains in certain stability properties of hyperbolic measures under perturbation: If μ is a hyperbolic measure for $\lim f_n$ then it is a weak limit of hyperbolic measures μ_n for the f_n (Corollary 5.8.5).

a. The Closing Lemma. The Closing Lemma for nonuniformly hyperbolic systems was originally proved in [**K3**]:

THEOREM 5.7.1. *Let M be a compact surface, $\alpha > 0$, $f \in \mathrm{Diff}^{1+\alpha}(M)$ and μ an f-invariant hyperbolic measure. Then $(\forall\delta > 0)(\exists\alpha > 0)(\forall h > 0)(\exists\beta > 0)(\forall m \in \mathbb{N})$: If $x, f^m(x) \in \Lambda_\delta$ with $d(x, f^m(x)) < \beta$ then there is a hyperbolic periodic point $z = f^m(z)$ whose stable and unstable manifolds are admissible $(s,\gamma,1)$- and $(u,\gamma,1)$-manifolds, respectively, for some $\gamma < 1$ and such that for some $\mu \in (0,1)$ and all $0 \le i \le m$ we have $d(f^i(z), f^i(x)) \le \alpha h \max\{\mu^{m-i}, \mu^i\}$.*

b. The Shadowing Lemma.

THEOREM 5.7.2 (Shadowing Lemma for nonuniformly hyperbolic systems). *Let M be a compact surface, $\alpha > 0$, $f \in \mathrm{Diff}^{1+\alpha}(M)$ and μ an f-invariant hyperbolic measure. For $\delta > 0$ there is a neighborhood $\widetilde{\Lambda}_\delta$ of Λ_δ such that for $\alpha > 0$ sufficiently small there is a $\beta > 0$ such that every β-pseudo-orbit $(x_m)_{m\in\mathbb{Z}}$ in $\widetilde{\Lambda}_\delta$ is α-shadowed by the orbit $\mathcal{O}(y)$ of some $y \in M$.*

SKETCH OF THE PROOF. For sufficiently small $\beta > 0$ consider a β-pseudo-orbit $(x_m)_{m\in\mathbb{N}}$ in $\widetilde{\Lambda}_\delta$. For some $\zeta < 1$ we inductively construct a $\zeta^k\beta$-pseudo-orbit $(x^k_m)_{m\in\mathbb{N}}$ for every $k \in \mathbb{N}$:

For $m \in \mathbb{Z}$ and $i = -1, 0, 1, 2$ choose $z_{m+i} \in \Lambda_\delta$ such that x_{m+i} lies in a Lyapunov chart around z_{m+i}. Choose an admissible (u,γ)-manifold V^u_m near z_{m-1} with $x_{m-1} \in V^u_m$, for γ small. Similarly choose an admissible (s,γ)-manifold V^s_m near z_{m+1} with $f(x_m) \in V^s_m$. If β is sufficiently small then $f(V^u_m)$ and $f^{-1}(V^s_m)$ are admissible manifolds near z_m, so let $x^1_m \in f(V^u_m) \cap f^{-1}(V^s_m)$. Likewise produce x^1_{m+1}. Now there is a $\zeta < 1$ such that $d(f(x_m^1), x_{m+1}^1) < \zeta d(f(x_m), x_{m+1})$. Iterate this construction starting with $\beta = (1-\zeta)\alpha$ and let $y = \lim_{k\to\infty} x_0^k$. Then

$$d(f^i(y), x_i) = \lim_{k\to\infty} d(f^i(y), x_i^k) \leq \beta/(1-\zeta) = \alpha.$$

□

c. The Lifschitz Theorem.

THEOREM 5.7.3 (Lifschitz Theorem for nonuniformly hyperbolic systems). *Let M be a compact Riemannian manifold, $\alpha > 0$, $f \in \mathrm{Diff}^{1+\alpha}(M)$, μ an f-invariant hyperbolic measure and $\varphi \colon M \to \mathbb{R}$ Hölder-continuous with $\sum_{i=0}^{m-1} \varphi(f^i(p)) = 0$ whenever $f^m(p) = p$. Then there is a measurable Borel function h such that $\varphi = h \circ f - h$ almost everywhere with respect to μ.*

REMARK. The proof only requires φ to be a Borel function whose restriction to Λ_δ is Hölder continuous with respect to a Lyapunov metric, with uniform Hölder exponent and constant for all $\delta > 0$. Note that the function h obtained in the theorem has the same property.

PROOF. Take $\delta > 0$ and suppose μ is ergodic. Then there exists $x \in \operatorname{supp}\mu$ such that $\operatorname{supp}\mu \subset \bar{\mathcal{O}}(x)$. Assume $x \in \Lambda_\delta$ and $\mathcal{O}(x) \cap \Lambda_\delta$ is dense in Λ_δ. Define $h(f^n(x)) = \sum_{i=0}^{n-1} \varphi(f^i(x))$. To extend h to Λ_δ continuously we need to show that h is uniformly continuous on $\mathcal{O}(x) \cap \Lambda_\delta$. Take $\alpha > 0$ and fix the corresponding β in the Closing Lemma. If $n_2 > n_1$ are such that $d(f^{n_2}(x), f^{n_1}(x)) < \beta$, then by the Closing Lemma there exists a $\mu < 1$ and a hyperbolic periodic point z satisfying $d(f^i(f^{n_1}(x)), f^i(z)) \leq \alpha \max\{\mu^{n_2-n_1-i}, \mu^i\}$ whenever $0 \leq i \leq n_2 - n_1 - 1$.

Now since φ is Hölder continuous, *i.e.*, $|\varphi(x) - \varphi(y)| \leq C\, d(x,y)^r$ for some $C > 0$ and $0 < r \leq 1$, we have

$$|\varphi(f^{n_1+i}(x)) - \varphi(f^i(z))| \leq C\alpha^r \max\{\mu^{r(n_2-n_1-i)}, \mu^{ri}\},$$

so

$$|h(f^{n_2}(x)) - h(f^{n_1}(x))| = \Big| \sum_{i=0}^{n_2-n_1-1} \big(\varphi(f^{n_1}(f^i(x))) - \varphi(f^i(z))\big) + \sum_{i=0}^{n_2-n_1-1} \varphi(f^i(z))\Big|$$

$$\leq C\alpha^r \sum_{i=0}^{n_2-n_1-1} \max\{\mu^{r(n_2-n_1-i)}, \mu^{ri}\} + \Big| \sum_{i=0}^{n_2-n_1-1} \varphi(f^i(z))\Big| \leq 2C\alpha^r \sum_{i=0}^{\infty} \mu^{ri} = \frac{2C\alpha^r}{1-\mu^r}.$$

Thus h is defined and continuous on Λ_δ. Extend h to $\bigcup_{i=0}^{\infty} f^i(\Lambda_\delta)$ as follows: On $f(\Lambda_\delta) \smallsetminus \Lambda_\delta$ set $h = \varphi \circ f^{-1} + h \circ f^{-1}$, and so on. Since $\mu\left(\bigcup_{i=0}^{\infty} f^i(\Lambda_\delta)\right) = 1$, h is defined almost everywhere and clearly satisfies the conditions of the theorem. □

REMARK. As in the uniformly hyperbolic case one can, in fact, show that the unstable distribution is Hölder continuous with respect to a Lyapunov metric on Λ_δ, similarly to the previous remark. Hence the restriction of the Jacobian to the unstable distribution is also Hölder continuous with respect to a Lyapunov metric. Indeed, the most important application of the Lifschitz Theorem concerns the logarithm of the unstable Jacobian.

8. Entropy and horseshoes

The theory also contains a beautiful result in line with our division into elliptic–parabolic and hyperbolic dynamical systems: The entropy of an ergodic hyperbolic measure, if positive, is approximated arbitrarily well by the topological entropies of horseshoes [**KH**]. In the case of surfaces, positive entropy of a measure implies hyperbolicity and hence by the Variational Principle the topological entropy is approximated by that of horseshoes. In other words, horseshoes are *the* mechanism for the production of exponential orbit growth. It should be noted that for interval maps the same happens even without smoothness and that in higher dimension the situation is entirely different. Entropy can be produced by other effects [**Hm**].

Recall that a compact f-invariant set Λ is a horseshoe for $f \in \mathrm{Diff}^1(M)$ if there exist $s, k \in \mathbb{N}$ and sets $\Lambda_0, \dots, \Lambda_{k-1}$ such that $\Lambda = \Lambda_0 \cup \dots \cup \Lambda_{k-1}$, $f^k(\Lambda_i) = \Lambda_i$, $f(\Lambda_i) = \Lambda_{i+1} \mod k$, and $f^k\restriction_{\Lambda_0}$ is conjugate to a full shift on s symbols. If $\dim M = 2$ and μ is an ergodic hyperbolic measure for $f \in \mathrm{Diff}(M)$ let $\chi(\mu) = \min\{|\chi_i| \mid i = 1, 2\}$. For a hyperbolic horseshoe Λ we can then define $\chi(\Lambda) = \inf\{\chi(\mu) \mid \mu$ is supported on a periodic orbit$\}$.

THEOREM 5.8.1. *Let M be a compact surface, $\alpha > 0$, $f \in \mathrm{Diff}^{1+\alpha}(M)$ and μ an ergodic f-invariant hyperbolic measure with $h_\mu(f) > 0$. Then for any $\rho > 0$ and any finite collection of functions $\varphi_1, \dots, \varphi_k \in C(M)$ there exists a hyperbolic horseshoe Λ satisfying the following conditions:*

1. $h_{top}(f\restriction_\Lambda) > h_\mu(f) - \rho$.
2. *Λ is contained in a ρ-neighborhood of* $\operatorname{supp} \mu$.
3. $\chi(\Lambda) > \chi(\mu) - \rho$.
4. *There exists a measure $\nu = \nu(\Lambda)$ supported on Λ such that for $i = 1, \dots, k$*

$$\Big| \int \varphi_i \, d\nu - \int \varphi_i \, d\mu \Big| < \rho.$$

Several easy consequences are worth noting.

COROLLARY 5.8.2. *For f and μ as in Theorem 5.8.1 there exists a sequence of f-invariant measures μ_n supported on hyperbolic horseshoes Λ_n such that*

1. *$\mu_n \to \mu$ in the weak* topology and*
2. $h_{\mu_n}(f) \to h_\mu(f)$.

The Variational Principle gives

COROLLARY 5.8.3. *Let M be a compact surface, $\alpha > 0$, $f \in \mathrm{Diff}^{1+\alpha}(M)$ with $h_{top}(f) > 0$. Then f has a hyperbolic periodic point.*

COROLLARY 5.8.4. *For f and μ as in Theorem 5.8.1 and $\epsilon > 0$*

$$h_\mu(f) \le \overline{\lim}_{m\to\infty} \frac{1}{m} \log \operatorname{card}\{x \in M \mid f^m(x) = x,\ \chi(x) \ge \chi(\mu) - \epsilon\}.$$

In particular $h_{top}(f) \le p(f)$, where $p(f)$ is the exponential growth rate of periodic points.

PROOF. If Λ is a hyperbolic horseshoe for f then

$$h_{\mathrm{top}}(f\restriction_\Lambda) = \overline{\lim}_{m\to\infty} \frac{1}{m} \log \operatorname{card}\{x \in \Lambda \mid f^m(x) = x\}.$$

Therefore by Theorem 5.8.1 the corollary follows. □

The following corollary shows that hyperbolic measures are "stable" or persistent under C^1 perturbations. This, of course, is a consequence of the structural stability of hyperbolic horseshoes.

COROLLARY 5.8.5. *Given f and μ as in Theorem 5.8.1 and $f_n \in \mathrm{Diff}^{1+\alpha}(M)$ such that $f_n \to f$ in the C^1 topology there exist f_n-invariant ergodic measures μ_n satisfying*

1. $\mu_n \to \mu$ *weakly,*
2. $h_{\mu_n}(f_n) \to h_\mu(f)$,
3. $\chi(\mu_n) \to \chi(\mu)$.

COROLLARY 5.8.6. *The entropy function $h\colon \mathrm{Diff}^{1+\alpha}(M^2) \to \mathbb{R}$ is lower semicontinuous.*

PROOF. By Theorem 5.8.1 $h(f) = \sup\{h(f\restriction_\Lambda) \mid \Lambda \text{ is a hyperbolic horseshoe}\}$. By structural stability of horseshoes lower semicontinuity follows. □

9. Sinai–Ruelle–Bowen measure

Because of its important role in the study of attractors, especially numerical experiments, there is great interest in producing a counterpart of the Sinai–Ruelle–Bowen measure outside the uniformly hyperbolic context.

In the nonuniformly hyperbolic case the definition of Sinai–Ruelle–Bowen measure is that it has absolutely continuous conditionals on unstable manifolds. This holds if and only if it satisfies the *Pesin entropy formula* [**Pe3**] $h_\mu = \int \sum \lambda_i^+ \dim E^i \, d\mu$ (where the λ_i^+ are the positive Lyapunov exponents and the E^i the corresponding subspaces) [**L2, LY**]. By the Ruelle entropy formula $h_\mu \le \int \sum \lambda_i^+ \dim E^i \, d\mu$ always. That absolute continuity of the Sinai–Ruelle–Bowen measure on unstable leaves entails the Pesin entropy formula [**LY**] can therefore be interpreted as saying that with respect to this measure entropy reflects all the expansion that happens in the system; there is no "wasted expansion".

In the nonuniformly hyperbolic case a Sinai–Ruelle–Bowen measure is also the asymptotic distribution for a set of points of positive Lebesgue measure [**PS3, L2**]. It also has a (possibly countable) spectral decomposition. This was proved by Pesin for the conservative case and by Ledrappier in the dissipative case [**L2**]. For this it is essential that no Lyapunov exponent is zero. Note the special case that a smooth invariant measure with no zero exponents is a Sinai–Ruelle–Bowen measure.

Existence of such a measure is harder to obtain [**HY, H**], despite some notable successes, such as with the Hénon attractor [**BY**]. An excellent survey is [**Yo**]. Simple examples suggest the difficulty. Smooth systems may fail to have a Sinai–Ruelle–Bowen measure even if hyperbolicity breaks down only in the most benign way. The example is a hyperbolic automorphism of $\mathbb{T}^2$ perturbed so as to remain hyperbolic except at the fixed point, where the derivative has an eigenvalue one and the other less than one [**HY**]. (This is also an example of nonuniqueness of equilibrium states [**K1**].) It is interesting that the introduction of benign singularities to the uniformly hyperbolic setting is not nearly as problematic [**Ch**]. When studying attractors, an essential problem is that sets of positive Lebesgue measure may have asymptotic distribution unrelated to the invariant measure of interest.

Nevertheless, there are some remarkable successes. First of all, the equivalence of the three characterizations of the Sinai–Ruelle–Bowen measure that constitute its main interest (equilibrium state, absolute continuity on unstable leaves, asymptotic distribution for Lebesgue-a.e. points, see Subsection 3.6e, [**S-BP**]), have a useful counterpart in the nonuniform situation. A measure satisfying Pesin's entropy formula [**S-BP**] (entropy is the integral of the positive Lyapunov exponents) is also absolutely continuous on unstable leaves and represents the asymptotic distribution of a set of points of positive Lebesgue measure [**L2**]. Therefore it is clear what to look for, and such a measure is again called a Sinai–Ruelle–Bowen measure.

The other success is that for some important attractors of nonuniform type, a Sinai–Ruelle–Bowen measure has been found. The Hénon attractor (for appropriate parameters) is the most prominent example [**BY**], and the techniques developed in that context have been greatly improved to cover an entire class of attractors [**WY**], specifically, attractors with one expanding direction that experience strong contraction in all other directions. This ongoing work of Wang and Young already gives a remarkably comprehensive and systematic framework for identifying hyperbolicity and stochastic properties.

10. Comparison

The list of structural results that transfer (with appropriate modification) from the uniform to the nonuniform situation is quite impressive, which can be taken as a testament to the basic robustness of the hyperbolic paradigm. Closing, shadowing, spectral decomposition, Markov partitions and absolute continuity remain valid with relatively moderate adjustment. Expansivity could be recovered in a substantially restated fashion that is hardly worthwhile. Ergodicity of volume is elusive as yet, but the counterpart is positive measure of ergodic components [**S-BP, P**].

More difficulties appear in conjunction with the theory of equilibrium states. Those with the Sinai–Ruelle–Bowen measure are a clear indication, but the recent successes are impressive. Remarkably, uniqueness (and ergodicity) of the measure of maximal entropy was proved recently for the case of geodesic flows on rank 1 (weakly hyperbolic) manifolds [**S-K**].

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