TOPOLOGICAL ENTROPY FOR NONUNIFORMLY CONTINUOUS MAPS

BORIS HASSELBLATT, ZBIGNIEW NITECKI, AND JAMES PROPP

Dedicated to Yakov Pesin on his 60th birthday

ABSTRACT. The literature contains several extensions of the standard definitions of topological entropy for a continuous self-map $f: X \to X$ from the case when X is a compact metric space to the case when X is allowed to be noncompact. These extensions all require the space X to be totally bounded, or equivalently to have a compact completion, and are invariants of uniform conjugacy. When the map f is uniformly continuous, it extends continuously to the completion, and the various notions of entropy reduce to the standard ones (applied to this extension). However, when uniform continuity is not assumed, these new quantities can differ. We consider extensions proposed by Bowen (maximizing over compact subsets and a definition of Hausdorff dimension type) and Friedland (using the compactification of the graph of f) as well as a straightforward extension of Bowen and Dinaburg's definition from the compact case, assuming that X is totally bounded, but not necessarily compact. This last extension agrees with Friedland's, and both dominate the one proposed by Bowen (Theorem 6). Examples show how varying the metric outside its uniform class can vary both quantities. The natural extension of Adler–Konheim–McAndrew's original (metric-free) definition of topological entropy beyond compact spaces dominates these other notions, and is unfortunately infinite for a great number of noncompact examples.

1. INTRODUCTION

There are two standard definitions of topological entropy for a continuous self-map of a compact metric space. The original definition by Adler, Konheim and McAndrew [AKM] via open covers applies to a continuous self-map of a compact topological space and is clearly an invariant of topological conjugacy. The reformulation of this definition by Bowen [B71] and Dinaburg [Din], based on the dispersion of orbits, requires a metric (or at least a uniform structure). For compact metric spaces these two definitions agree. In particular, the Bowen–Dinaburg version of entropy is independent of the (compact) metric used to compute it.

When X is not compact, the situation is more complicated. The aforementioned entropy notions extend somewhat straightforwardly to this context, but these extensions may no longer agree, and other natural extensions may yield

Date: June 9, 2007.

2000 *Mathematics Subject Classification.* 37B40.

Key words and phrases. topological entropy, totally bounded metric space, compactification, nonuniform continuity.

yet different values. In [B71], Bowen proposed an invariant based on measuring the dispersion of orbits emanating from a compact subset $K \subset X$ and taking the supremum over all such subsets K. Bowen was motivated by uniformly continuous examples, and his definition has been taken as a "standard" definition of topological entropy for a (uniformly continuous) self-map of a metric space which is not assumed to be compact (see, for example, [Wal, pp. 168–176]). We shall formulate this in Section 3 and call it *Bowen compacta entropy* and denote it by h_{Bc}.

In [B73] Bowen defined entropy directly for noncompact spaces, based on ideas related to Hausdorff dimension. We denote this notion by h_{BH} but do not consider this in depth here except to note a pertinent observation of his.

However, in many cases of interest, such as billiards on tables with corners or meromorphic self-maps on complex projective spaces, X occurs naturally as a subset of a compact metrizable ambient space, giving a preferred class of metrics, but the map is not uniformly continuous in this metric. Friedland [F91], motivated by examples of the second type, started from an interpretation of the Bowen–Dinaburg calculation (in the compact case) by Gromov [Gro], and proposed a different invariant, based on a compactification of the graph of the map (Subsection 2.2). We call this the *Friedland entropy* and denote it by h_F.

Both of these invariants begin from a metric which, as we shall see, must have a compact completion; both are unchanged if the metric is replaced by a uniformly equivalent one (so that the completions are homeomorphic) but both can change if the new metric is equivalent, but not uniformly equivalent, to the original one.

In this note we approach this situation abstractly and intrinsically. We start with a continuous self-map $f\colon X \to X$, assuming that X is a (not necessarily closed) subset of a compact metric space. Thus, the restriction of the ambient metric to X is *totally bounded*, i.e., for every $\varepsilon > 0$ there exists a finite cover of X by balls of radius ε. However, the map f is not assumed to be uniformly continuous, and hence need not extend continuously to the (compact) closure of X in the ambient space. Our main observation is that the Bowen–Dinaburg calculation can be used *verbatim* in this context[1], and the resulting invariant h_{BD} coincides with that defined by Friedland, which in turn dominates the invariant proposed by Bowen, while all of these are dominated by the natural extension of the Adler–Konheim–McAndrew entropy h_{AKM}:

Theorem 1.1. *If (X,d) is a totally bounded metric space and $f\colon (X,d) \to (X,d)$ continuous then $h_{AKM}(f) \geq h_F(f) = h_{BD}(f,X,d) \geq h_{Bc}(f,X,d)$ and $h_{AKM}(f) \geq h_{BH}(f)$. If f is a homeomorphism then also $h_{BD}(f,X,d) = h_{BD}(f^{-1},X,d)$.*

After posting an earlier version of this paper on the arXiv, we learned from Vincent Guedj that he had recently published a proof of the equality $h_F(f) = h_{BD}(f)$ [Gu, Lemme 1.1]; we thank him for pointing this out to us.

[1]Such a procedure is followed tacitly in, for example, [DS].

Theorem 1.2. *Both inequalities in Theorem 1.1 may be strict even for totally bounded spaces. Indeed, there is a map* f *such that for any* $h \geq 0$ *there is a metric* d *with* $h_{BD}(f, X, d) = h_{BD}(f^{-1}, X, d) = h_{Bc}(f^{-1}, X, d) = h$ *while* $h_{AKM}(f) = \infty$ *and* $h_{Bc}(f, X, d) = 0$.

By Theorem 1.1 the Friedland–Bowen–Dinaburg entropy of a homeomorphism equals that of its inverse, but Theorem 1.2 shows that this may fail in an extreme way for Bowen compacta entropy. Theorem 1.2 is a consequence of the more explicit and comprehensive Proposition 5.1 below.

Theorem 1.2 shows in particular that $h_{AKM}(f)$ may often be infinite, and $h_{BD}(f, X, d)$ can be made large. This situation does not just arise for select examples but whenever the dynamics is "essentially noncompact":

Theorem 1.3. *Suppose* X *is a topological space and* $f\colon X \to X$ *is continuous and has an orbit that* escapes to infinity *in the sense that it eventually leaves every compact subset, or equivalently, has no accumulation points (i.e.,* $\omega(x) = \emptyset$*). Then*

(1) *If* X *carries a totally bounded metric and* $N \in \mathbb{N}$ *then there is a totally bounded metric* d_N *on* X *such that* $h_{BD}(f, X, d_N) \geq N$.
(2) $h_{AKM}(f) = \infty$*, whether or not* X *carries a totally bounded metric.*

In Section 2 we review various definitions of entropy for continuous maps of compact spaces and make some observations about these that are useful in Section 3, where we exhibit some extensions to noncompact situations. Section 4 establishes the relations in Theorem 1.1, while Section 5 establishes the phenomena described by Theorem 1.2 and Theorem 1.3.

2. Entropy in Compact Spaces

We briefly review the definitions of Adler–Konheim–McAndrew and Bowen–Dinaburg.

2.1. **Adler–Konheim–McAndrew's Definition.** In [AKM], the topological entropy of a continuous self-map $f\colon X \to X$ of a compact topological space is defined as follows. Given an open cover α of X, denote by $H(\alpha)$ the cardinality of a minimal subcover. The entropy of f relative to α is

$$h(f, \alpha) := GR\{H(\alpha_n)\}, \tag{2.1}$$

where $\alpha_n := \{\bigcap_{i=0}^{n-1} f^{-i}(O_i) \mid O_0, \dots, O_{n-1} \in \alpha\}$ and $GR\{c_n\} := \overline{\lim}_{n\to\infty} \frac{1}{n} \log c_n$ is the (exponential) growth rate.

Definition 2.1. If X is compact then the *Adler–Konheim–McAndrew entropy* of $f\colon X \to X$ is $h_{AKM}(f) := \sup\{h(f, \alpha) \mid \alpha \text{ is a finite open cover of } X\}$.

Finiteness of the covers is, of course, irrelevant in a compact space, but allows us to remove the compactness assumption in this definition.

Since this definition is purely topological we obviously have

Proposition 2.2. $h_{AKM}(f)$ *is an invariant of topological conjugacy.*

2.2. **Bowen–Dinaburg entropy.** Given a metric space (X,d), we say that a subset $S\subset X$ ε*-separated* with respect to d for some $\varepsilon>0$ if distinct points of S are spaced at least ε apart ($s\neq s'\in S\Rightarrow d(s,s')\geq\varepsilon$), and that it ε*-spans* a subset $K\subset X$ if every point of K is within distance ε of some point of S ($\forall x\in K\ \exists s\in S$ such that $d(x,s)<\varepsilon$). When $S\subset K$, we can say S is "ε-dense" in K.

Lemma 2.3. *If there exist finite ε-spanning sets for X with respect to d then*

$$\operatorname{minspan}[K,d,\varepsilon]:=\min\{\operatorname{card}S\,|\,S\subset X\ \varepsilon\textit{-spans } K \textit{ with respect to } d\}<\infty$$

$$\operatorname{maxsep}[K,d,\varepsilon]:=\max\{\operatorname{card}S\,|\,S\subset K \textit{ is } \varepsilon\textit{-separated with respect to } d\}<\infty$$

$$\operatorname{minspan}[K,d,\varepsilon]\leq\operatorname{maxsep}[K,d,\varepsilon]\leq\operatorname{minspan}[K,d,\varepsilon/2]. \tag{2.2}$$

Proof. An ε-separated set which is maximal with respect to inclusion is also ε-dense in X. The second inequality in (2.2) is an easy application of the triangle inequality [B71, Lemma 1]. □

If $f\colon X\to X$ is continuous and $n\in\mathbb{N}$ then the *Bowen–Dinaburg metric* is

$$d_n^f(x,x'):=\max_{0\leq i<n} d(f^i(x),f^i(x')). \tag{2.3}$$

We say that a set (n,ε)*-spans* $K\subset X$ (resp. is (n,ε)*-separated*) if it ε-spans K (resp. is ε-separated) with respect to d_n^f.

If d is compact then so is d_n^f for $n\in\mathbb{N}$ and Lemma 2.3 applies.

Definition 2.4. If (X,d) is compact and $f\colon X\to X$ continuous then the *Bowen–Dinaburg entropy* of f on $K\subset X$ is

$$\begin{aligned} h_{BD}(f,K,d)&:=\lim_{\varepsilon\to0} GR\{\operatorname{maxsep}[K,d_n^f,\varepsilon]\}\\ &=\lim_{\varepsilon\to0} GR\{\operatorname{minspan}[K,d_n^f,\varepsilon]\}. \end{aligned} \tag{2.4}$$

Remark 2.5 ([Wal, Chap. 7])**.** If (X,d) is compact then $h_{BD}(f,X,d)=h_{AKM}(f)$.

2.3. **Gromov's Observation.** Gromov [Gro] gave an alternate description of the Bowen–Dinaburg entropy, which we present here in slightly different language than that used in [Gro]. It motivated Friedland's extension to noncompact spaces (Definition 3.5).

The set of sequences in a topological space X is the product

$$X^{\mathbb{N}}:=\prod_{i=0}^{\infty}X=\{\mathbf{x}:=x_0x_1\ldots\,|\,x_i\in X\}$$

with the product topology. If X is compact then so is $X^{\mathbb{N}}$, and if (X,d) is a bounded metric space then

$$\hat{d}(\mathbf{x},\mathbf{x}'):=\sum_{i=0}^{\infty}\rho^{-i}d(x_i,x_i') \tag{2.5}$$

defines a metric on $X^{\mathbb{N}}$ for any $\rho>1$. The *shift* $\sigma\colon X^{\mathbb{N}}\to X^{\mathbb{N}}$, $\sigma(\mathbf{x})_i:=x_{i+1}$ ($i=0,1,\ldots$) is continuous and preserves

$$X^f:=\{\mathbf{x}=x_0,x_1,\ldots\,|\,x_{i+1}=f(x_i)\text{ for } i=0,1,\ldots\}. \tag{2.6}$$

Proposition 2.6. *$f: X \to X$ and $\sigma \restriction X^f$ have the same Bowen–Dinaburg entropy.*

Proof. The map $x \mapsto \mathbf{x} := (x, f(x), f^2(x), \dots)$ conjugates $f: X \to X$ and $\sigma \restriction X^f$. □

The topological entropy of a subshift of $\sigma: (1, \dots, N-1)^{\mathbb{N}} \to (1, \dots, N-1)^{\mathbb{N}}$ is the growth rate of the number of words of length n. The entropy of σ on X^f can be defined analogously.

Given $\varepsilon > 0$, $n \in \mathbb{N}$ and ε-balls $B_0, \dots, B_{n-1} \subset X$, we define an *$\varepsilon$-cube of order n* by $\{\mathbf{x} \in X^{\mathbb{N}} \mid x_i \in B_i \text{ for } i = 0, \dots, n-1\}$. The *$\varepsilon$-capacity* $\mathrm{cap}[X^f, n]$ of X^f of order n is the minimum number of ε-cubes of order n needed to cover X^f.

Proposition 2.7 ([Gro])**.** $h_{top}(\sigma \restriction X^f) = \lim_{\varepsilon \to 0} GR\{\mathrm{cap}[X^f, n]\}$.

3. Definitions of Entropy in Noncompact Spaces

How can the preceding definitions be adapted to $f: X \to X$ when X is *not* compact?

3.1. **Bowen Compacta Entropy.** Bowen's first approach to this question [B71] was to note that Definition 2.4 only requires K (rather than X) to be compact and hence the entropy of f on any compact $K \subset X$ is still well-defined.

Remark 3.1. Here K is *not* assumed to be f-invariant. We are thus measuring the dispersion of orbits *emanating from* K, but not confined to it, and hence even at this level, the entropy of f "on" a compact set K can—and does—depend on the choice of metric we use to calculate it.

Definition 3.2 (See [B71])**.** If X is a metric space and $f: X \to X$ continuous then the *Bowen compacta entropy* of f is

$$h_{Bc}(f, X, d) := \sup\{h_{BD}(f, K, d) \mid K \subset X \text{ compact}\}.$$

Remark 3.3. Bowen made the assumption of uniform continuity in this definition since his motivating examples were all uniformly continuous. He proved that $h_{Bc}(f, X, d)$ behaves as expected under uniform equivalence of metrics, iteration and cartesian products [B71, Propositions 3, 4]. In fact, when f is uniformly continuous, then it (uniquely) extends continuously to the completion of (X, d), which is compact if (X, d) is totally bounded. By Corollary 4.6 $h_{Bc}(f, X, d)$ is then obtained from any of the definitions for the compact case. Accordingly, we avoid the assumption of uniform continuity.

Note that Theorem 1.2 and examples outlined by Walters [Wal, p. 176] show that without uniform continuity a number of useful properties of entropy that hold for uniformly continuous maps can fail.

3.2. **Friedland Entropy.** Rational maps on complex projective space $\mathbb{CP}^n$ are defined only on a subset of the compact space $\mathbb{CP}^n$, and are definitely not uniformly continuous. To define topological entropy for such maps, Friedland [F91, F95] adapted Gromov's point of view.

Remark 3.4 ([Kel, p. 198], [Mun, p. 276])**.** (X, d) is totally bounded if and only if the completion $(\hat{X}, \hat{d})$ of (X, d) is compact.

Thus, if (X,d) is totally bounded and $f\colon X\to X$ is continuous then $\hat{X}^{\mathbb{N}}$ is compact and $X^f\subset\hat{X}^{\mathbb{N}}$ (defined as in (2.6)) is invariant under the shift $\sigma\colon\hat{X}^{\mathbb{N}}\to\hat{X}^{\mathbb{N}}$.

Definition 3.5. If (X,d) is totally bounded and $f\colon X\to X$ continuous then the *Friedland entropy* of f is $h_F(f):=h_{top}(\sigma\,|\,\mathrm{clos}\;X^f)$.

Note that we do not assume f to be uniformly continuous.

Friedland's definition does not explicitly involve a metric, but rather an embedding of the space X in some compact topological space. Of course, if X carries a totally bounded metric, it singles out such an embedding. It should also be noted that Friedland used this formulation to extend the notion of entropy beyond iterated mappings, to more general relations [F96].

3.3. **Adler–Konheim–McAndrew and Bowen–Dinaburg Entropies.** As we immediately noted, the definition of the Adler–Konheim–McAndrew entropy (Definition 2.1) makes sense as stated for noncompact spaces, without any additional assumptions on the map.

If we assume total boundedness of X—which is equivalent to finiteness of $\mathrm{minspan}[X,d,\varepsilon]$ and $\mathrm{maxsep}[X,d,\varepsilon]$ for all $\varepsilon>0$—then by Lemma 3.7 below the Bowen–Dinaburg entropy also extends straightforwardly:

Remark 3.6. Definition 2.4 makes sense if (X,d) is a totally bounded metric space, $f\colon X\to X$ continuous (not necessarily uniformly) and $K\subset X$. We are interested primarily in the case $K=X$.

Lemma 3.7. *If (X,d) is totally bounded, then so are the Bowen–Dinaburg metrics d_n^f. In particular, the numbers* $\mathrm{minspan}[X,d_n^f,\varepsilon]$ *and* $\mathrm{maxsep}[X,d_n^f,\varepsilon]$ *are finite for $n=0,1,2,\dots$ and $\varepsilon>0$.*

Proof. The completion of the maximum metric on X^n

$$d_{max}((x_1,\dots,x_n),(x_1',\dots,x_n')):=\max_{i=1,\dots,n}d(x_i,x_i')$$

is the maximum metric on $\hat{X}^n$. The latter is compact, and contains the collection of orbit segments $(x,f(x),\dots,f^{n-1}(x))$. □

Proposition 3.8. *If (X,d) is totally bounded then $h_{BD}(f,X,d)\le h_{AKM}(f)$.*

Proof. Given $\varepsilon>0$ consider a minimal cover α by ε-balls. If $n\in\mathbb{N}$ and S is (n,ε)-separated then $H(\alpha_n)\ge\mathrm{card}\,S$ because each element of α_n contains at most one element of S. □

If h is a homeomorphism and S is (n,ε)-separated under h then $h^n(S)$ is (n,ε)-separated under h^{-1}. This gives the last sentence of Theorem 1.1:

Proposition 3.9. *If (X,d) is totally bounded and $f\colon X\to X$ a homeomorphism then $h_{BD}(f,X,d)=h_{BD}(f^{-1},X,d)$.*

3.4. **Bowen's Hausdorff entropy.** We reproduce verbatim Bowen's definition in [B73] of entropy in the form of a Hausdorff dimension, where the "size" of a set is determined by how the map acts on it, rather than by its diameter. This definition does not require total boundedness or uniform continuity.

Suppose X is a metric space and $f\colon X\to X$ is continuous. If α is a finite open cover of X then for $E\subset X$ we write $E\prec\alpha$ if $E\subset A\in\alpha$ for some A, and for $\mathscr{E}\subset\mathscr{P}(X)$ we write $\mathscr{E}\prec\alpha$ if $E\prec\alpha$ for every $E\in\mathscr{E}$. Define $n_{f,\alpha}(E):=0$ if $E\not\prec\alpha$, $n_{f,\alpha}(E):=+\infty$ if $f^k(E)\prec\alpha$ for all $k\in\mathbb{N}$ and

$$n_{f,\alpha}(E):=\max\{n\in\mathbb{N}\colon f^k(E)\prec\alpha \text{ for } 0\le k\le n\}\quad\text{otherwise.}$$

Then

$$D_\alpha(E):=e^{-n_{f,\alpha}(E)}\quad\text{and}\quad D_\alpha(\{E_i\}_{i\in\mathbb{N}},\lambda):=\sum_{i\in\mathbb{N}}D_\alpha(E_i)^\lambda$$

for $\lambda\in\mathbb{R}$ and

$$m_{\alpha,\lambda}(Y):=\lim_{\epsilon\to 0}\inf\{D_\alpha(\mathscr{E},\lambda)\colon Y\subset\bigcup\mathscr{E}\text{ and }D_\alpha(E)<\epsilon\text{ for }E\in\mathscr{E}\}$$

defines a measure such that $m_{\alpha,\lambda}(Y)\le m_{\alpha,\lambda'}(Y)$ when $\lambda'<\lambda$ and $m_{\alpha,\lambda}(Y)\notin\{0,+\infty\}$ for at most one $\lambda\in\mathbb{R}$. Accordingly, $h_\alpha(f,Y):=\inf\{\lambda\colon m_{\alpha,\lambda}(Y)=0\}$ is well-defined.

Definition 3.10. $h_{BH}(f,Y):=\sup\{h_\alpha(f,Y)\colon \alpha \text{ a finite open cover}\}$ is called the *Bowen Hausdorff entropy* of f on Y, and $h_{BH}(f):=h_{BH}(f,X)$ is called the *Bowen Hausdorff entropy* of f.

Remark 3.11. Bowen immediately proves that in the compact case this agrees with the Adler–Konheim–McAndrew entropy [B73, Proposition 1]. We presented his notion here for completeness and to observe that the first part of Bowen's proof of this does not use compactness and hence establishes that $h_{BH}\le h_{AKM}$, as recorded in Theorem 1.1.

Proof that $h_{BH}\le h_{AKM}$ [B73, p. 126]. We use the notations of Subsection 2.1. If α is a finite open cover of X and $\mathscr{E}_n$ a minimal subcover of α_n then $D_{\alpha,\lambda}(\mathscr{E}_n,\lambda)\le H(\alpha_n)e^{-n\lambda}$ and

$$m_{\alpha,\lambda}(X)\le\lim_{n\to\infty}(\exp(-\lambda+n^{-1}\log H(\alpha_n)))^n\le 0$$

if $\lambda>GR\{H(\alpha_n)\}=h(f,\alpha)$, so $h_\alpha(f,X)\le h(f,\alpha)$. □

3.5. **Coding of interval-exchange transformations.** In some contexts, there is a natural way of replacing a noncompact dynamical system by a symbolic system, that is to say a subshift of the full shift over some finite alphabet. Interval-exchange transformations provide a simple example. Consider for instance the interval exchange on the open interval $(0,1)$ that maps $(0,\alpha)$ affinely to $(1-\alpha,1)$ and $(\alpha,1)$ affinely to $(0,1-\alpha)$, with α irrational. (Maps like this arise as Poincaré sections for the billiards flow on a compact table with internal corners; the endpoints of the intervals being exchanged correspond to orbit-segments that hit a corner.) This map is not defined at points x in $(0,1)$ that are fractional parts of some multiple of α; therefore, if we want to view the dynamics as arising from iteration of a function from a domain to itself, that domain should be $(0,1)$ minus a countable dense set of points. Call this noncompact domain X. We can map our dynamical system on X into the 2-shift by partitioning X into $X \cap (0,\alpha)$ and $X \cap (\alpha,1)$ and then coding an orbit in X by a binary string in the usual way. Let ψ denote the map from X to the 2-shift. When we take the closure of $\psi(X)$, we add countably many limit points, obtaining a compact shift-invariant set X'. If one can show that the ergodic nonatomic shift-invariant measures on X' are all supported on $\psi(X)$ and hence correspond to the ergodic nonatomic invariant measures on X (as is the case for systems derived from polygonal billiards [Kat]), then one may feel justified in regarding the two systems as closely linked and defining the entropy of the former to equal the entropy of the latter. We mention this approach to compactifying dynamical systems, but we will not pursue it beyond suggesting that the affinity with Friedland's approach ought to be explored further.

4. Relation between Entropies and proof of Theorem 1.1

We need to address two issues with respect to the entropies introduced in Section 3. First, we need to establish the extent to which they are invariants, and second, we need to establish the relations between them.

As noted in Section 2, when X is compact, the Bowen Compacta entropy, the Friedland entropy and the Bowen–Dinaburg entropy agree with the Adler–Konheim–McAndrew entropy, which implies topological invariance by Proposition 2.2. However, in the context of totally bounded spaces, $h_{AKM}(f)$ does not in general agree with any of the other entropies (Section 5), so we need to attack the invariance question differently.

Definition 4.1. Suppose $f\colon (X,d)\to(X,d)$ and $\tilde f\colon (\tilde X,\tilde d)\to(\tilde X,\tilde d)$ are (not necessarily uniformly) continuous self-maps of totally bounded metric spaces. A *semiconjugacy*[2] from $\tilde f\colon (\tilde X,\tilde d)\to(\tilde X,\tilde d)$ to $f\colon (X,d)\to(X,d)$ is a continuous surjection $h\colon (\tilde X,\tilde d)\to(X,d)$ satisfying $h\circ\tilde f = f\circ h$; it is a *uniform semiconjugacy* if h is uniformly continuous with respect to the metrics $\tilde d$ and d: that is, for each $\varepsilon>0$, there exists $\tilde\varepsilon>0$ such that $\tilde d(x,x')<\tilde\varepsilon$ implies $d(h(x),h(x'))<\varepsilon$. The choice of $\tilde\varepsilon$ given ε is *modulus of continuity* for h. If h also has a uniformly continuous inverse then it is a *uniform conjugacy.*

Lemma 4.2. *Suppose $(\tilde X,\tilde d)$ and (X,d) are totally bounded, $\tilde f\colon (\tilde X,\tilde d)\to(\tilde X,\tilde d)$, $f\colon (X,d)\to(X,d)$ are continuous, and $h\colon (\tilde X,\tilde d)\to(X,d)$ is a uniform semiconjugacy.*

(1) *h is uniformly continuous with respect to corresponding pairs of Bowen–Dinaburg metrics, with the same modulus of continuity as with respect to the metrics $\tilde d$ and d.*

[2] Sometimes f is referred to as a *factor* of $\tilde f$ in this case.

(2) *If* $n \in \mathbb{N}$ *and* $\varepsilon > 0$ *then*

$$\operatorname{maxsep}[\tilde{X}, \tilde{d}_n^{\tilde{f}}, \tilde{\varepsilon}] \geq \operatorname{maxsep}[X, d_n^f, \varepsilon] \tag{4.1}$$

$$\operatorname{minspan}[\tilde{X}, \tilde{d}_n^{\tilde{f}}, \tilde{\varepsilon}] \geq \operatorname{minspan}[X, d_n^f, \varepsilon]. \tag{4.2}$$

Proof. (1): Let ε and $\tilde{\varepsilon}$ be related as in the definition of uniform continuity for h, as above. Note that if $\tilde{d}_n^{\tilde{f}}(x, x') < \tilde{\varepsilon}$ then $\tilde{d}(\tilde{f}^i(x), \tilde{f}^i(x')) < \tilde{\varepsilon}$ for each $i = 0, \ldots, n-1$, which implies (for each i) that $d(h(\tilde{f}^i(x)), h(\tilde{f}^i(x'))) < \varepsilon$, but since $h \circ \tilde{f}^i = f^i \circ h$, this is the same as $d_n^f(h(x), h(x')) < \varepsilon$.

(2): If $S \subset X$ is ε-separated with respect to d_n^f, form $\tilde{S} \subset \tilde{X}$ by picking a single preimage of each element of S. Then $\tilde{S}$ must be $\tilde{\varepsilon}$-separated with respect to $\tilde{d}_n^{\tilde{f}}$, by part (1), and has the same cardinality as S, giving (4.1).

If $\tilde{S} \subset \tilde{X}$ is $\tilde{\varepsilon}$-spanning with respect to $\tilde{d}_n^{\tilde{f}}$, then $S = h(\tilde{S}) \subset X$ is ε-spanning with respect to d_n^f, since (by surjectivity of h) given $x \in X$ we can pick $\tilde{x} \in \tilde{X}$ with $h(\tilde{x}) = x$, and then pick $\tilde{s} \in \tilde{S}$ with $\tilde{d}_n^{\tilde{f}}(\tilde{x}, \tilde{s}) < \tilde{\varepsilon}$, and it follows that $s = h(\tilde{s}) \in S$ satisfies $d_n^f(x, s) < \varepsilon$. This gives (4.2). □

Corollary 4.3. *In the situation of Lemma 4.2*

$$h_{BD}(\tilde{f}, \tilde{X}, \tilde{d}) \geq h_{BD}(f, X, d), \tag{4.3}$$

so h_{BD} *is an invariant of uniform conjugacy.*

This also goes a long way towards showing that Bowen compacta entropy is an invariant of uniform conjugacy; however, to obtain the corresponding analogue of (4.3) we need to know that the preimage of every compact subset $K \subset X$ is a compact subset of $\tilde{X}$—that is, the semiconjugacy must be *proper*:

Proposition 4.4. *If* $h\colon (\tilde{X}, \tilde{d}) \to (X, d)$ *is a proper, uniform semiconjugacy from* $\tilde{f}\colon \tilde{X} \to \tilde{X}$ *to* $f\colon X \to X$*, then* $h_{Bc}(\tilde{f}, \tilde{X}, \tilde{d}) \geq h_{Bc}(f, X, d)$.

We could establish directly that Friedland entropy is an invariant of uniform conjugacy, but this is an immediate corollary of Theorem 1.1. Aside from Remark 3.11 the proof of Theorem 1.1 is based on two observations. The first observation is

Lemma 4.5. *Suppose* (X, d) *is totally bounded and* $Y \subset X$ *is dense.*

(1) *If* $0 < \varepsilon' < \varepsilon$ *then* $\operatorname{maxsep}[Y, d, \varepsilon'] \geq \operatorname{maxsep}[X, d, \varepsilon]$.
(2) *If* $\varepsilon' > \varepsilon > 0$ *then* $\operatorname{minspan}[Y, d, \varepsilon'] \leq \operatorname{minspan}[X, d, \varepsilon]$.

Proof. (1): Suppose $S \subset X$ is ε-separated. Given $0 < \varepsilon' < \varepsilon$, let $\delta = \frac{1}{2}(\varepsilon - \varepsilon') > 0$. For each $s \in S$, pick $y \in Y$ with $d(s, y) < \delta$, and let $S' \subset Y$ be the resulting set of y's. For $s \neq s' \in S$, let y, y' be the corresponding points in S'. Then the triangle inequality gives

$$d(y, y') \geq d(s, s') - [d(s, y) + d(s', y')] > \varepsilon - 2\delta = \varepsilon',$$

so $S' \subset Y$ is ε'-separated and has cardinality the same as S.

(2): Suppose $S \subset X$ ε-spans X. Given $\varepsilon' > \varepsilon$, let $\delta = \varepsilon' - \varepsilon$, and for each $s \in S$ pick $s' \in Y$ with $d(s', s) < \delta$. Again, form the set $S' \subset Y$ of all such s''s. If $y \in Y \subset X$, we can find $s \in S$ with $d(s, y) < \varepsilon$; then

$$d(s', y) \le d(s', s) + d(s, y) < \delta + \varepsilon = \varepsilon'.$$

Thus, $S' \subset Y$ is a ε'-spanning subset of Y with cardinality at most that of S. □

Corollary 4.6. *If (X, d) is totally bounded, $f: (X, d) \to (X, d)$ is continuous and $Y \subset X$ is dense then $h_{BD}(f, X, d) = h_{BD}(f, Y, d)$.*

The conjugacy $\varphi: X \to X^f$, $x \mapsto \mathbf{x} := (x, f(x), f^2(x), \ldots)$ from the proof of Proposition 2.6 that takes $x \in X$ to its orbit has as its inverse $\pi \restriction X^f$, where $\pi: X^{\mathbb{N}} \to X$, $(x_0, x_1, \ldots) \mapsto x_0$ is the projection to the first factor of $X^{\mathbb{N}}$.

Lemma 4.7. (1) *If $n \in \mathbb{N}$ then $\pi: (X^f, \hat{d}) \to (X, d_n^f)$ is uniformly continuous.*

(2) *For each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$,*

$$d_n^f(x, x') < \varepsilon \Rightarrow \hat{d}(\varphi(x), \varphi(x')) < (N+1)\varepsilon.$$

Proof. (1) If $\rho > 1$ is as in (2.5), $\hat{d}(x, x') < \varepsilon' := \rho^{-n}\varepsilon$ and $j < n$ then

$$\rho^{-j} d(f^j(x), f^j(x')) \le \sum_{i=0}^{\infty} \rho^{-i} d(f^i(x), f^i(x')) = \hat{d}(x, x') < \varepsilon',$$

so $d(f^j(x), f^j(x')) < \rho^j \varepsilon' < \varepsilon$. Maximizing over $j < n$ gives $d_n^f(x, x') < \varepsilon$.

(2) Since X is totally bounded, it has finite diameter, so given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\sum_{i=N}^{\infty} \rho^{-i} < \varepsilon / \operatorname{diam}(X, d).$$

If $n \ge N$ and $d_n^f(x, x') < \varepsilon$ then

$$\begin{aligned}
\hat{d}(\varphi(x), \varphi(x')) &= \sum_{i=0}^{N-1} \rho^{-i} d(f^i(x), f^i(x')) + \sum_{i=N}^{\infty} \rho^{-i} d(f^i(x), f^i(x')) \\
&\le N \max_{i<N} d(f^i(x), f^i(x')) + \sum_{i=N}^{\infty} \rho^{-i} \operatorname{diam}(X, d) \\
&\le N d_n^f(x, x') + \varepsilon \\
&< (N+1)\varepsilon.
\end{aligned}$$

□

Proof of Theorem 1.1. The first inequality is Proposition 3.8.

To establish the equality between Friedland entropy and Bowen–Dinaburg entropy first use Lemma 4.2, Lemma 4.7 (1) and Corollary 4.6 to get

$$h_{BD}(f, X, d) \le h_{BD}(\sigma, X^f, \hat{d}) = h_{BD}(\sigma, \operatorname{clos} X^f, \hat{d}) = h_F(f).$$

In the other direction, given $\varepsilon > 0$ pick N as in Lemma 4.7 (2) and $n > N$. If $S \subset X$ is $\varepsilon/(N+1)$-spanning with respect to d_n^f then $\varphi(S)$ ε-spans X^f with respect to $\hat{d}$, so

$$\operatorname{minspan}[X, d_n^f, \frac{\varepsilon}{N+1}] \ge \operatorname{minspan}[X^f, \hat{d}, \varepsilon],$$

hence

$$GR\{\text{minspan}[X, d_n^f, \frac{\varepsilon}{N+1}]\} \geq GR\{\text{minspan}[X^f, \hat{d}, \varepsilon]\}.$$

Letting $\varepsilon \to 0$ we get $h_{BD}(f, X, d) \geq h_F(f)$.

Finally, to show that Bowen–Dinaburg entropy dominates Bowen compacta entropy, note that the latter is the supremum of Bowen–Dinaburg entropy over the collection of compact subsets of X and that the Bowen–Dinaburg entropy of the restriction of a map to a subset is at most that of the map on the ambient space. □

5. Dependence on the uniform structure—Theorems 1.2 and 1.3

The original definition of Adler–Konheim–McAndrew made no reference to a metric structure, but was formulated in the context of compact topological spaces. It is tempting, therefore, to adopt Definition 2.1 as a definition of topological entropy (that is, using only *finite* open covers, even if the space is not compact). While this conjugacy invariant is independent of any metric or uniform structure, the content of Theorem 1.3 is that it is unfortunately also infinite for most essentially noncompact examples.

Proof of Theorem 1.3. (1) Suppose there is a totally bounded metric d on X, or equivalently, that X is embeddable in a compact metric space. Fix N, U_i, and s_i as above. Using the Urysohn Lemma, let ϕ be a continuous bounded function supported inside the neighborhoods U_i of the orbit such that

$$\phi(f^i(x)) = s_i + 1, \quad i = 0, 1, \ldots.$$

Then X embeds in $X \times \mathbb{R}$ as the graph of ϕ, and we can use the natural product metric, $\tilde{d}$, which is totally bounded on the graph. It is easy to see that if $s_i \neq s_j$ then $\tilde{d}(f^{s_i}(x), f^{s_j}(x)) \geq 1$; from this it follows that by picking a point of the orbit in each of the sets $A_w \in \alpha_n$ we obtain an $(n, 1)$-separated set of cardinality N^n, and the rest follows by standard arguments.

(2) When X is totally bounded, this is an obvious consequence of Proposition 3.8 and (1), but to obtain this claim without assuming total boundedness a direct proof is needed.

If the orbit $\{f^i(x)\}_{i=0}^{\infty}$ of x has no accumulation points then it is a closed, countable and discrete set. In particular, $V := X \setminus \{f^i(x) \mid i = 0, 1, \ldots\}$ is open. Pick disjoint open neighborhoods U_i, $i = 0, 1, \ldots$ of $f^i(x)$.

Given $N \in \mathbb{N}$, let $\{s_i\}_{i=0}^{\infty}$ be a sequence of integers between 0 and $N-1$ ($s_i \in \{0, \ldots, N-1\}$ for all i) containing every finite "word" $w = w_0 \ldots w_{n-1}$, in the sense that for some i_w, $s_{i_w+j} = w_j$, $i = 0, \ldots, N-1$. Now, group the neighborhoods U_i of $f^i(x)$ according to the values of s_i: for $k = 1, \ldots, N-1$, define

$$A_k := \bigcup \{U_i \mid s_i = k\}, \quad k = 1, \ldots, N-1$$

and define A_0 similarly, but adjoining V

$$A_0 := V \cup \bigcup \{U_i \mid s_i = 0\}.$$

Then

$$\alpha := \{A_0, \dots, A_{N-1}\}$$

is a cover of X by N open sets. It has no proper subcover, because each $f^i(x)$ belongs only to A_{s_i}. Furthermore, if for each "word" $w = w_0 \dots w_{n-1}$ we define an element of α_n (see Subsection 2.1) by

$$A_w := A_{w_0} \cap f^{-1}(A_{w_1}) \cap \dots \cap f^{-(n+1)}(A_{w_{n-1}})$$

then $f^{i_w}(x)$ belongs to A_w and to none of the sets corresponding to other words of length n. This shows that the refinement α_n also has no proper subcovers; thus $\log H(\alpha_n) = n \log N$, and $h(f, \alpha) = \log N$, so the supremum $h_{AKM}(f)$ over all finite covers is infinite. □

It is easy to see that a similar phenomenon occurs if some point x_0 "escapes to infinity" in backward time, in the sense that there exists a sequence of successive preimages x_{-i}, $i = 0, 1, \dots$, ($f(x_{-i}) = x_{1-i}$ for $i \ge 1$) with no accumulation points.

Theorem 1.3 shows that we cannot hope to avoid the effects of a choice of metric on the definition of topological entropy in a noncompact setting. An elaboration of the technique of proof for this result can be used to illustrate the ways that our two metric notions of entropy—as well as the relation between them—can be affected by the choice of metric.

Proof of Theorem 1.2. In general, one way to define a totally bounded metric on a topological space X is to (topologically) embed X in a compact metric space; then the restriction of the metric to the embedded image of X defines a metric on X which is clearly totally bounded. We use this trick to prove Theorem 1.2. Specifically, we show the following:

Proposition 5.1. *Suppose* $f\colon (0,1) \to (0,1)$ *is a homeomorphism with* $f(x) > x$ *for all* $x \in (0,1)$. *Then for* $H \ge 1$ *there is a totally bounded metric* d_H *on* $(0,1)$ *such that*

$$h_{BD}(f, (0,1), d_H) = h_{BD}(f^{-1}, (0,1), d_H) = h_{Bc}(f, (0,1), d_H) = \log H \tag{5.1}$$

and

$$h_{Bc}(f^{-1}, (0,1), d_H) = 0, \tag{5.2}$$

while $h_{AKM}(f) = h_{AKM}(f^{-1}) = \infty$ *and* $h_\mu(f) = 0$ *for any invariant Borel probability measure* μ.

We note in passing that this result is more general than may at first appear, because of the following easy observation [KH, Proof of Proposition 2.1.7].

Remark 5.2. Any two fixedpoint-free self-homeomorphisms of open intervals are topologically conjugate.

In particular, f and f^{-1} are topologically conjugate, and hence the metric for which $h_{Bc}(f, (0,1), d) = \log N$ and $h_{Bc}(f^{-1}, (0,1), d_H) = 0$ can be turned into a metric d for which $h_{Bc}(f,(0,1),d) = 0$ and $h_{Bc}(f^{-1},(0,1),d) = \log N$ by composing the embedding defining d_H with the homeomorphism that conjugates f with f^{-1}.

Proof of Proposition 5.1. We first take care of the special case $H = 1$. In the standard metric on $(0,1)$, f extends to the closed interval by fixing the endpoints. This extension has topological (hence Bowen–Dinaburg) entropy zero, since the topological entropy of any continuous self-map of a compact space equals the entropy of its restriction to its nonwandering set ([B70, Xio]), which in this case consists of the two endpoints. But then Corollary 4.6 tells us that the same holds for $f\colon (0,1)\to(0,1)$. Moreover, this proves the later claim that $h_\mu(f) = 0$ for every invariant Borel probability measure μ. Also, $h_{AKM}(f) = h_{AKM}(f^{-1}) = \infty$ is obtained from Theorem 1.3 or Proposition 3.8.

For $H > 1$, we consider the map $f_H\colon \mathbb{R}\to\mathbb{R}$, $x\mapsto Hx$, which is conjugate to f by the composition of log, $\tan^{-1}$ and the conjugacy from Remark 5.2. We use the metric d_{cyl} on $\mathbb{R}^+$ induced by the Euclidean metric on $\mathbb{R}\times\mathbb{C}$ by the embedding

$$S\colon \mathbb{R}^+ \hookrightarrow (0,1)\times S^1 \subset \mathbb{R}\times\mathbb{C}, \quad x\mapsto \Big(\frac{1}{1+x}, e^{2\pi i x}\Big).$$

Note that $d_{\mathbb{R}}(x,y) \le 1/2$ implies

$$4d_R(x,y) \le d_{cyl}(x,y) \le \sqrt{(2\pi)^2+1}\, d_{\mathbb{R}}(x,y) < 7d_{\mathbb{R}}(x,y).$$

This shows that if $\epsilon > 0$ is small enough (such as $\epsilon < 1/10H$) then S maps a set that is ϵ/H^n-separated in $[1/10,1/2]$ for $d_{\mathbb{R}}$ to one that is ϵ-separated for $(d_{\mathbb{R}})^f_n$. Since there are such sets of cardinality H^n/ϵ, this implies $h_{BD}(f,(0,1),d_H) \ge h_{Bc}(f,(0,1),d_H) \ge \log H$.

To see the reverse inequality, take M such that $S(\mathbb{R}^+)$ is contained in an $\epsilon/2$-neighborhood of $S(0,M)$. Then S maps any $\epsilon/14H^n$-spanning set in $(0,M)$ to a subset of $S(0,\infty)$ that is ϵ-spanning for $(d_{\mathbb{R}})^f_n$. Since one can find such sets of cardinality $14(M+1)H^n/\epsilon$, this gives $h_{Bc}(f,(0,1),d_H) \le h_{BD}(f,(0,1),d_H) \le \log H$.

To see (5.2) take K compact, $\epsilon > 0$ and let $N\in\mathbb{N}$ be such that $\operatorname{diam} f^{-1}(K) < \epsilon$. Then the number of (n,ϵ)-separated sets is independent of n for $n \ge N$ and hence has growth rate zero. □

Other examples, showing that Bowen compacta entropy for a nonuniformly continuous map fails to enjoy several useful properties of topological entropy on compact spaces (including that noted above) are outlined by Walters [Wal, p. 176].

References

[AKM] Roy L. Adler, A. G. Konheim, and M. H. McAndrew, *Topological entropy*, Transactions, American Mathematical Society **114** (1965), 309–319.

[ALM] Lluís Alsedà, Jaume Llibre, and Michał Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, second ed., Advanced Series in Nonlinear Dynamics, vol. 5, World Scientific, 2000.

[B70] Rufus Bowen, *Topological entropy and Axiom A*, Global Analysis, Proc. Symp. Pure Math, vol. 14, American Mathematical Society, 1970, pp. 23–42.

[B71] ——, *Entropy for group endomorphisms and homogeneous spaces*, Transactions, American Mathematical Society **153** (1971), 401–414, erratum, **181**(1973) 509-510.

[B73] ——, *Topological entropy for noncompact sets*, Transactions, American Mathematical Society **184** (1973), 125–136.

[Din] E. I. Dinaburg, *The relation between topological entropy and metric entropy*, Soviet Math. Dokl. **11** (1970), 13–16.

[DS] Tien-Cuong Dinh and Nessim Sibony, *Une borne supérieure pour l'entropie topologique d'une application rationnelle*, `arXiv:math.DES/0303271`.

[F91] Shmuel Friedland, *Entropy of polynomial and rational maps*, Annals of Mathematics **133** (1991), 359–368.

[F95] ——, *Entropy of algebraic maps*, J. Fourier Analysis and Appl. (1995), 215–228, (Kahane Special Issue).

[F96] ——, *Entropy of graphs, semigroups and groups*, Ergodic Theory of $\mathbb{Z}^d$-Actions (M. Pollicott & K. Schmidt, ed.), London Math. Soc. Lecture Notes Series, vol. 228, Cambridge Univ. Press, 1996, pp. 319–343.

[Gro] Mikhail Gromov, *On the entropy of holomorphic maps*, L'Enseignment Mathématique **49** (2003), 217–235.

[Gu] Vincent Guedj, *Entropie topologique des applications méromorphes*, Ergodic Theory and Dynamical Systems 25 (2005), 1847–1855.

[Kat] Anatole Katok, *The growth rate for the number of singular and periodic orbits for a polygonal billiard*, Communications in Mathematical Physics **111** (1987), 151–160.

[KH] Anatole Katok, Boris Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995

[Kel] John L. Kelley, *General topology*, D. van Nostrand, 1955.

[MS] Michał Misiurewicz and Wiesław Szlenk, *Entropy of piecewise monotone mappings*, Studia Mathematica **67** (1980), 45–63.

[Mun] James R. Munkres, *Topology*, second ed., Prentice-Hall, 2000.

[Wal] Peter Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, no. 79, Springer, New York and Berlin, 1982.

[Xio] Jincheng Xiong, *A note on topological entropy*, Chinese Science Bulletin **34** (1989), 1673–6.

DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, MEDFORD, MA 02155
E-mail address: `boris.hasselblatt@tufts.edu`

DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, MEDFORD, MA 02155
E-mail address: `zbigniew.nitecki@tufts.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WI 53706
E-mail address: `propp@math.wisc.edu`