DEGREE-GROWTH OF MONOMIAL MAPS

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ABSTRACT. For projectivizations of rational maps Bellon and Viallet defined the notion of algebraic entropy using the exponential growth rate of the degrees of iterates. We want to call this notion to the attention of dynamicists by computing algebraic entropy for certain rational maps of projective spaces (Theorem 6.2) and comparing it with topological entropy (Theorem 5.1). The particular rational maps we study are monomial maps (Definition 1.2), which are closely related to toral endomorphisms. Theorems 5.1 and 6.2 imply that the algebraic entropy of a monomial map is always bounded above by its topological entropy, and that the inequality is strict if the defining matrix has more than one eigenvalue outside the unit circle. Also, Bellon and Viallet conjectured that the algebraic entropy of every rational map is the logarithm of an algebraic integer, and Theorem 6.2 establishes this for monomial maps. However, a simple example using a monomial map shows that a stronger conjecture of Bellon and Viallet is incorrect, in that the sequence of algebraic degrees of the iterates of a rational map of projective space need not satisfy a linear recurrence relation with constant coefficients.

1. INTRODUCTION

1.1. **Algebraic entropy.** In their 1998 paper [BV], Bellon and Viallet introduced the concept of "algebraic entropy" for the study of iterates of rational maps, measuring the rate at which the algebraic degree of the Nth iterate of the map grows as a function of N. This natural and appealing notion (foreshadowed in work of Arnold [Ar] and paralleled in work by Russakovskii and Shiffman [RS], albeit with different terminology) seems to have escaped the attention of most researchers in ergodic theory and dynamical systems; to our knowledge, the only articles on this topic that have appeared in *Ergodic Theory and Dynamical Systems* thus far are [Ma] and [Gu]. Hence, a major motivation behind the writing of this article is a desire to advertise the study of degree-growth and to encourage readers of this journal to think about transporting established ideas from measurable and topological dynamics into the setting of algebraic geometry. More specifically, the following conjecture deserves attention from dynamicists of an algebraic bent:

Conjecture 1.1 (Bellon and Viallet)**.** *The algebraic entropy of every rational map is the logarithm of an algebraic integer.*

1.2. **Monomial maps and projectivization.** A second purpose in writing this article is to show that a simple class of rational maps provides insight into fundamental questions about algebraic entropy.

Definition 1.2. Every n-by-n nonsingular integer matrix $A = (a_{ij})_{i,j=1}^n$ determines a mapping $(x_1,\dots,x_n) \mapsto (y_1,\dots,y_n)$ from a dense open subset U of complex n-space $\mathbb{C}^n$ to itself by

$$y_i = \prod_j x_j^{a_{ij}}.$$

(If all $a_{ij} \geq 0$, then $U = \mathbb{C}^n$.) We call this an *affine monomial map.*

Remark 1.3. The map carries the n-torus $\{(x_1,\dots,x_n) : |x_1| = \cdots = |x_n| = 1\}$ to itself, and the restriction of the map to the n-torus is isomorphic to the toral endomorphism associated with A.

In this article we focus on a slightly different construction, namely, the projectivization of the affine monomial map. Each projectivized monomial map sends a certain dense open subset U of complex projective n-space $\mathbf{CP}^n$ to itself. (See Section 2 for relevant definitions and notation.) Moreover, the action of the map on the n-torus $\{(x_1 : \dots : x_{n+1}) : |x_1| = \cdots = |x_{n+1}| \neq 0\} \subseteq \mathbf{CP}^n$ is once again isomorphic to the toral endomorphism associated with A.

Remark 1.4. In accordance with algebraic geometry nomenclature, we refer to maps from $\mathbb{C}^n$ to itself as "affine" and maps from $\mathbf{CP}^n$ to itself as "projective".

Example 1.5. Let A be the 1-by-1 matrix whose sole entry is 2. The affine monomial map associated with A is the squaring map $z \mapsto z^2$ on $\mathbb{C}$, whereas the projective monomial map associated with A is the squaring map on the complex projective line $\mathbf{CP}^1$, also known as the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Example 1.5 is atypical in that the squaring map is well-defined on all of $\mathbf{CP}^1$. Later we will see for most integer matrices A we need to restrict the monomial map associated with A to a dense open proper subset U of $\mathbf{CP}^n$. (From here on, the term "monomial map" will usually refer to a complex projective monomial map unless otherwise specified.)

1.3. **Relations between entropies.** A monomial map restricted to U is continuous, so it makes sense to ask about its topological entropy. Since U is typically not a compact space, it is not immediately clear how the topological entropy should be defined; fortunately, [HNP] shows that some of the most natural candidate definitions agree and clarifies the relation between the main notions that have been proposed. In Section 5, we show that for this notion of topological entropy, the topological entropy of the monomial map associated with the matrix A is no less than the topological entropy of the toral endomorphism associated with A, which in turn is equal to the logarithm of the product of $|z|$ as z ranges over all the eigenvalues of A outside the unit circle (Theorem 5.1).

At the same time, monomial maps fall into the framework of Bellon and Viallet, and we show (Theorem 6.2) that the algebraic entropy of a monomial map is equal to the logarithm of the spectral radius of the associated n-by-n integer matrix, i.e., the maximum value of the logarithm of $|z|$ as z ranges over all the eigenvalues of A.

Theorems 5.1 and 6.2 imply that the algebraic entropy of a monomial map does not exceed its topological entropy, and that the inequality is strict if the defining matrix has more than one eigenvalue outside the unit circle.

Since the entries of A are integers, the eigenvalues of A are all algebraic integers. Thus Theorem 6.2 (or, rather, Corollary 6.4) provides support for the Bellon–Viallet Conjecture 1.1. On the other hand, we devise a monomial map that falsifies a stronger conjecture of Bellon and Viallet's, namely, that the sequence of degrees of the iterates of a rational map satisfy a linear recurrence with constant coefficients. The trick is to choose a matrix A whose dominant eigenvalues are a pair of complex numbers $re^{i\theta}$, $re^{-i\theta}$ where θ is incommensurable with 2π. For such an A, the sequence of degrees is a patchwork of a finite collection of integer sequences that individually satisfy linear recurrences with constant coefficients; the degree sequence jumps around between elements of the family in a nonperiodic fashion. Details are given in Section 7.

We also describe in Section 8 an analogue of algebraic entropy applicable to the dynamics of piecewise linear maps.

These discoveries are not deep; they illustrate that there is a lot of "low-hanging fruit" in the study of iteration of rational maps from a projective space to itself, and suggest that a more vibrant interaction between the dynamical systems community and the integrable systems community (perhaps mediated by researchers in the field of several complex variables) could lead to more rapid progress in the development of the theory of algebraic dynamical systems.

2. DEFINITIONS

We review some basic facts about projective geometry (more details can be found in [Mu]) before commencing a discussion of algebraic degree and algebraic entropy (drawing heavily on [BV]).

2.1. **Projective space.**

Definition 2.1. Complex projective n-space is defined as $\mathbf{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where $u \sim v$ iff $v = cu$ for some $c \in \mathbb{C} \setminus \{0\}$. We write the equivalence class of $(x_1, x_2, \ldots, x_{n+1})$ in $\mathbf{CP}^n$ as $(x_1 : x_2 : \ldots : x_{n+1})$.

The standard embedding $\mathscr{P} : (x_1, x_2, \ldots, x_n) \mapsto (x_1 : x_2 : \ldots : x_n : 1)$ of affine n-space into projective n-space has an "inverse map" $\mathscr{A} : (x_1 : \ldots : x_n : x_{n+1}) \mapsto (\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}})$. The ratios x_i / x_{n+1} $(1 \le i \le n)$, defined on a dense open subset of $\mathbf{CP}^n$, are the *affine coordinate variables* on $\mathbf{CP}^n$.

Geometrically, one may model $\mathbf{CP}^n$ as the set of lines through the origin in $(n+1)$-space. In this model, the point $(a_1 : a_2 : \ldots : a_{n+1})$ in $\mathbf{CP}^n$ corresponds to the line $x_1/a_1 = x_2/a_2 = \cdots = x_{n+1}/a_{n+1}$ in $\mathbb{C}^{n+1}$ (for those i with $a_i = 0$, we impose the condition $x_i = 0$). The intersection of this line with the hyperplane $x_{n+1} = 1$ is the point

$$(\frac{a_1}{a_{n+1}}, \frac{a_2}{a_{n+1}}, \ldots, \frac{a_n}{a_{n+1}}, 1)$$

(as long as $a_{n+1} \neq 0$). We identify affine n-space with the hyperplane $x_{n+1} = 1$. Affine n-space in this way becomes a Zariski-dense subset of projective n-space. (See e.g., [Ha] for the definition and basic properties of the Zariski topology.) Since there is nothing special about the $n+1$st coordinate in $\mathbf{CP}^n$, each of the hyperplanes $x_i = 1$ $(1 \le i \le n+1)$ is a copy of (complex) affine n-space. Thus we might see projective n-space as the result of gluing together $n+1$ affine n-spaces in a particular way. Under this viewpoint, a monomial map is the result of gluing together $n+1$ compatible toral endomorphisms in a particular way.

Definition 2.2. We define the distance between two points in $\mathbf{CP}^n$ as the angle $0 \le \theta \le \pi/2$ between the lines in $\mathbb{C}^{n+1}$ associated with those points; this gives a metric on $\mathbf{CP}^n$, and the resulting metric topology coincides with the quotient topology on $(\mathbb{C}^{n+1} \setminus \{0\})/\sim$.

Remark 2.3. There is a more natural distance on projective space, namely the distance induced by the Riemannian "Fubini–Study metric", and it may play a role in the analysis of the topological entropy of monomial maps; however, we will not pursue this topic here.

2.2. **Rational maps and projectivization.** We will use the term *rational map* in two different ways: both to refer to a function from (a Zariski-dense subset of) $\mathbb{C}^n$ to $\mathbb{C}^m$ given by m rational functions of the affine coordinate variables, and to refer to the associated function from a Zariski-dense subset of $\mathbf{CP}^n$ to $\mathbf{CP}^m$. (Henceforth, we will refer to rational maps "from $\mathbb{C}^n$ to $\mathbb{C}^m$" or "from $\mathbf{CP}^n$ to $\mathbf{CP}^m$", even though the map may be undefined on a proper subvariety of the domain.) That is, a "rational map" may be affine or projective, according to context.

Definition 2.4. The *projectivization* of an affine map f is the map $\mathscr{P} \circ f \circ \mathscr{A}$, written with cleared fractions.

Example 2.5. The partial function $f\colon x \mapsto 1/x$ on affine 1-space (undefined at $x = 0$) is associated with the function $g\colon (x : y) \mapsto (y : x)$ on projective 1-space (defined everywhere). f is its own inverse on its domain, while g is its own inverse globally.

The maps in this and the next example (some of them partial functions from affine n-space to itself and some of them partial functions from projective n-space to itself) will be called rational maps, and the context should make it clear whether we are in the affine setting or the projective setting. In both settings, we identify functions that agree on a Zariski-dense set. Under this identification, projectivization commutes with composition, so, in particular, the Nth power of the projectivization of an affine map is identified with the projectivization of the Nth power of the map.

Example 2.6. The partial function $f\colon (x, y) \mapsto (1/x, 1/y)$ on affine 2-space (undefined at $xy = 0$) is associated with the function $g\colon (x : y : z) \mapsto (yz : xz : xy)$ on projective 2-space. g is undefined on $xy = xz = yz = 0$, and its composition with itself is undefined on the proper subvariety $xyz = 0$ and is the identity map elsewhere. With the above identification we can say that $g \circ g$ is the identity map and say that g is self-inverse.

Definition 2.7. A *birational (projective) map* is a rational map f from $\mathbf{CP}^n$ to $\mathbf{CP}^n$ with a rational inverse g (satisfying $f \circ g = g \circ f =$ the identity map on a Zariski-dense subset of $\mathbf{CP}^n$).

Example 2.8. The affine map $(x, y) \mapsto (y, xy)$ with inverse $(x, y) \mapsto (y/x, x)$ projectivizes as $f\colon (x : y : z) \mapsto (yz : xy : z^2)$ with inverse $g\colon (x : y : z) \mapsto (yz : x^2 : xz)$. (As a check, note that $f(g(x : y : z)) = ((x^2)(xz) : (yz)(x^2) : (xz)^2) = (x : y : z)$.)

2.3. **Degree.**

Lemma 2.9. *Every rational map from* $\mathbf{CP}^n$ *to* $\mathbf{CP}^m$ *can be written in the form* $(x_1 : \dots : x_{n+1}) \mapsto (p_1(x_1, \dots, x_{n+1}) : \dots : p_{m+1}(x_1, \dots, x_{n+1}))$ *where the* $m+1$ *polynomials* $p_1, \dots, p_{m+1}$ *are homogeneous polynomials of the same degree having no joint common factor.*

Proof. When we apply $\mathscr{A}$, we get n ratios of the affine coordinate variables, with each ratio homogeneous of degree 0. When we then apply f, we get m rational functions of the affine coordinate variables, with each rational function homogeneous of degree 0, and when we apply $\mathscr{P}$, we tack on a 1 at the end of the n-tuple, obtaining an $(n+1)$-tuple. When we clear denominators, we multiply all $n+1$ of the rational functions of degree 0 by some homogeneous polynomial, and when we remove common factors, we divide them by some homogeneous polynomial. The end result is an $(n+1)$-tuple of homogeneous polynomials of the same degree, having no joint common factor (although any proper subset of the polynomials may have some factor in common). □

Definition 2.10. The common degree of the polynomials in Lemma 2.9 is called the *degree* of the map.

Example 2.11. The most familiar case is $n = 1$: the rational function $x \mapsto p(x)/q(x)$ (where p and q are polynomials with no common factor) is associated with the projective map $(x:y) = (x/y:1) \mapsto (p(x/y)/q(x/y):1) = (p(x/y):q(x/y))$. The rational functions $p(x/y)$ and $q(x/y)$ are homogeneous of degree 0; to make them polynomials in x and y, we must multiply through by $y^{\max(\deg p, \deg q)}$. Hence the degree of the mapping is $\max(\deg p, \deg q)$.

Example 2.12. A simple example with $n > 1$ is given by the projectivization of the monomial map $(x, y) \mapsto (y, xy)$ of Example 2.8. The map $f\colon (x:y:z) \mapsto (yz:xy:z^2)$ is of degree 2, and its square $f^2 = f \circ f\colon (x:y:z) \mapsto ((xy)(z^2):(yz)(xy):(z^2)^2) = (xyz^2:xy^2z:z^4) = (xyz:xy^2:z^3)$ is of degree 3.

Example 2.13. More generally, a 2-by-2 nonsingular integer matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is associated with the affine map $(x, y) \mapsto (x^a y^b, x^c y^d)$ and with the projective map $(x:y:z) = (x/z:y/z:1) \mapsto ((x/z)^a(y/z)^b:(x/z)^c(y/z)^d:1)$. To make all three entries monomials in x, y, and z, we multiply them by $x^{\max(-a,-c,0)}$, $y^{\max(-b,-d,0)}$ and $z^{\max(a+b,c+d,0)}$, so the degree of the mapping is $\max(-a,-c,0) + \max(-b,-d,0) + \max(a+b, c+d, 0)$. Applying this to the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

reproduces the calculations of the preceding example.

More generally still, we have:

Proposition 2.14. *If A is an n-by-n nonsingular matrix with integer entries a_{ij}, the degree of the projective map associated with A is equal to*

$$D(A) := \sum_{j=1}^{n} \mathrm{Max}_{i=1}^{n}(-a_{ij}) + \mathrm{Max}_{i=1}^{n}\Big(\sum_{j=1}^{n} a_{ij}\Big), \tag{2.1}$$

where $\mathrm{Max}(\dots) := \max(0, \dots)$.

For each fixed n, the function $D(\cdot)$, viewed as a function on the space of all real n-by-n matrices, is continuous and piecewise linear. That is, the hyperplanes given by all the equations $a_{ij} = 0$ $(1 \le i, j \le n)$, $a_{ij} = a_{i'j}$ $(1 \le i, i', j \le n)$, $\sum_{j=1}^{n} a_{ij} = 0$ $(1 \le i \le n)$, and $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{i'j}$ $(1 \le i, i' \le n)$ yield a decomposition of $\mathbb{R}^{n^2}$ into chambers such that for all A within each closed chamber C, we have $D(A) = L_C(A)$ for some linear map $L_C\colon \mathbb{R}^{n^2} \mapsto \mathbb{R}$. Indeed, the degree of the monomial map associated with A is precisely $\max_C L_C(A)$, where C varies over all the chambers.

Example 2.15 (Degree and birational conjugacy)**.** If we conjugate the involution $(x, y) \mapsto (y, x)$ via the birational involution $(x, y) \mapsto (x, x^2 - y)$, we get the involution $(x, y) \mapsto (x^2 - y, (x^2 - y)^2 - x)$. When we projectivize, we get a map $(x : y : z) \mapsto (x^2z^2 - yz^3 : (x^2 - yz)^2 - xz^3 : z^4)$ of degree 4 that is conjugate to the map $(x : y : z) \mapsto (y : x : z)$ of degree 1. This demonstrates the important point that the degree of a projective map is not invariant under birational conjugacy. However, as we will see in the next subsection, the rate at which the degree of a projective map grows under iteration of the map *is* invariant under birational conjugacy.

2.4. **Algebraic entropy.** Bellon and Viallet's notion of algebraic entropy, like most notions of entropy, owes its existence to an underlying subadditivity/submultiplicativity property:

$$\deg(f \circ g) \leq \deg(f)\deg(g) \tag{2.2}$$

for all rational maps f, g. This is an easy consequence of Lemma 2.9; strict inequality in the lemma holds precisely when the compositions of the polynomials have some factors in common. This is the "reduction-of-degree" phenomenon.

A first consequence of this inequality (via a standard argument; e.g., Proposition 9.6.4 of [KH]) is that $(1/N)\log\deg(f^N)$ converges to a limit, that is, algebraic entropy is well-defined:

Definition 2.16. $\lim_{N\to\infty}(1/N)\log\deg(f^N)$ is called the (Bellon–Viallet) *algebraic entropy* of f.

A second consequence of (2.2), no less important, is that if $g = \phi^{-1} \circ f \circ \phi$ for some birational ϕ, then f and g have the same algebraic entropy.

Proposition 2.17. *Algebraic entropy is invariant under birational conjugacy.*

Remark 2.18. It should be mentioned that another use of the term "algebraic entropy" occurs in the dynamical systems literature, measuring the growth of complexity of elements of a finitely generated group under iteration of some endomorphism of the group; see, e.g., Definition 3.1.9 in [KH] and the recent article [Os]. There does not appear to be any connection between these two uses of the phrase.

3. Existing literature

Bellon and Viallet's definition arose from a large body of work in the integrable systems community on the issue of degree-growth; see, e.g., [FV], [HV1] and [HV2]. More recent articles on the topic coming from this community include [Be], [LRGOT] and [RGLO].

3.1. **Dynamical degrees.** A notion equivalent to Bellon and Viallet's was introduced at the same time in independent work by Russakovskii and Shiffman [RS], drawing upon earlier work by Friedland and Milnor [FM]. Russakovskii and Shiffman's theory associates various quantities, called dynamical degrees, with a rational map; the algebraic entropy is simply the logarithm of the dynamical degree of order 1. To give the flavor of this work (without purporting to define the notions being used), we state that the kth dynamical degree of a rational map f from $\mathbf{CP}^n$ to itself is given by

$$\lim_{N\to\infty}\left(\int (f^N)^*(\omega^k)\wedge\omega^{n-k}\right)^{1/N}$$

where ω denotes a Kähler form on $\mathbf{CP}^n$ (a complex (1,1) form).

3.2. **Intersections.** Algebraic entropy has antecedents elsewhere in dynamics. For, as was pointed out by Bellon and Viallet, the degree of a map is equal to the number of intersections between the image of a generic line in $\mathbf{CP}^n$ and a generic hyperplane in $\mathbf{CP}^n$. Thus algebraic entropy measures the growth rate of the number of intersections between one submanifold and the image of another submanifold, and is therefore related to the intersection-complexity research program of Arnold [Ar], introduced in the early 1990s and mostly neglected since then by mathematicians (though studied by some physicists: see e.g., [BM] and [AABM]). The intermediate dynamical degrees of

Russakovskii and Shiffman can be given definitions in this framework; specifically, the kth dynamical degree of a rational map from $\mathbf{CP}^n$ to itself (for any k between 0 and n) is equal to the number of intersections between the image of a generic $\mathbf{CP}^k$ in $\mathbf{CP}^n$ and a generic $\mathbf{CP}^{n-k}$ in $\mathbf{CP}^n$. Taking $k = n$, we see that the top dynamical degree of a rational map is precisely its topological degree (the number of preimages of a generic point).

It is worth remarking that some articles (such as [BM] and [AABM]), in keeping with Arnold's terminology, use the term "complexity" of f to refer to $\lim_{N\to\infty} \sqrt[N]{\deg(f^N)}$, so complexity is just another name for dynamical degree of order 1.

More recent articles on the topic of dynamical degree, intersections, and algebraic entropy include [BK], [BFJ], [DF], [DS], [FJ], [T1], [T2], and [TEGORS]. These articles often employ the language of several complex variables, with the apparatus of de Rham currents and cohomology. See also Friedland's survey [Fr4].

Lastly, we mention Veselov's survey article [V], which contains a good treatment of the multifarious notion of integrability.

4. Examples

In this section we present a collection of examples, making some basic observations about most of them. Several of these examples appear repeatedly in later sections to illustrate salient points at appropriate times.

Example 4.1. The Hénon map $(x, y) \mapsto (1 + y - Ax^2, Bx)$ projectivizes as $(x : y : z) \mapsto (z^2 + yz - Ax^2 : Bxz : z^2)$. For any nonzero constants A and B, the Nth iterate of this map has degree 2^N, so every nondegenerate Hénon map has algebraic entropy $\log 2$. This important example is discussed in detail by Bellon and Viallet.

Example 4.2. The map $f : (x, y) \mapsto (y, (y^2 + 1)/x)$ is the composition of the two involutions $(x, y) \mapsto ((y^2 + 1)/x, y)$ and $(x, y) \mapsto (y, x)$ but is itself of infinite order. Its projectivization is the map $(x : y : z) \mapsto (xy : y^2 + z^2 : xz)$. It can be shown that the degree of f^N is only $2N$. Hence the algebraic entropy of f is zero. This example is discussed in greater depth in [MP], [Ze], and [Ho2]. (Amusingly, if one replaces y^2 by y in the definition of the affine map f, one obtains a map of order 5 that was probably known to Gauss because of its connection with his *pentagramma mirificum* and is described in some detail in [FR].)

Example 4.3 (Somos-4 recurrence)**.** The map $(w, x, y, z) \mapsto (x, y, z, (xz + y^2)/w)$ has a similar flavor. Its Nth iterate has degree that grows like N^2, so it too has algebraic entropy zero. This is the Somos-4 recurrence, introduced by Michael Somos and first described in print by David Gale [Ga].

Remark 4.4 (Laurent phenomenon)**.** In the two preceding examples, the iterates of the map are all Laurent polynomials (rational functions that can be written as a polynomial divided by a monomial) thanks to "fortuitous" cancellations that occur every time one performs a division that a priori might be expected to yield a denominator with more than one term. (For Example 4.2, a proof of "Laurentness" can be found in [SZ]; for Example 4.3, see [FZ, Theorem 1.8].) Fomin and Zelevinsky call this the "Laurent phenomenon". For instance, in the case of example Example 4.2, the iterates of the (affine) map involve rational functions of x and y with denominators x, x^2y, x^3y^2, x^4y^3, etc., even though a priori one would expect denominators with two or more terms to arise. Specifically, (x, y) gets mapped to $(y, (y^2 + 1)/x)$, which gets mapped to $((y^2 + 1)/x, (y^4 + 2y^2 + 1 + x^2)/x^2y$, and so on. Indeed, when one iterates f complicated denominators do arise, but they always disappear when one cancels common factors between numerators and denominators. E.g., when one squares $(y^4+2y^2+1+x^2)/x^2y$, adds 1, and divides by $(y^2+1)/x$, one expects to see a factor of y^2+1 in the denominator, but the numerator turns out to contain a factor of y^2+1 as well, so that the end result simplifies to the Laurent polynomial $(y^6+3y^4+3y^2+2x^2y^2+x^4+2x^2+1)/x^3y^2$. In the projective context, this simplification turns into an instance of the reduction-of-degree phenomenon alluded to in Section 2.4. Thus the Laurent phenomenon can be seen as an important case of the reduction-of-degree phenomenon, where reduction-of-degree applies in a significant way to all the iterates of the map. The Laurent phenomenon has strong connections to the confinement-of-singularities phenomenon (see e.g. [GRP], [HV1], [HV2], [LRGOT], and [T1]).

Example 4.5. The map $f\colon (w,x,y,z)\mapsto (x,y,z,z(wz-xy)/(wy-x^2))$ does not quite fall under the heading of the Laurent phenomenon, but comes close. In the iterates of this map, the denominators are always a power of $xz-y^2$ times a power of $wy-x^2$. The degrees of these iterates are 3, 5, 9, 13, 17, 23, 29, 37, 45, 53, 63, 73, 85, 97, ... This unfamiliar-looking sequence is actually five quadratic sequences patched together: $\deg(f^N)=(2/5)N^2+(6/5)N+c_N$, where the c_N depend only on the residue class of N modulo 5. (Indeed, $\deg(f^N)=\lfloor(2N^2+6N+9)/5\rfloor$; this formula was guessed by us and proved by A. Hone in private correspondence.) Once again, the algebraic entropy is zero.

Example 4.6 (The Scott map)**.** The map $f\colon (x,y,z)\mapsto (y,z,(y^2+z^2)/x)$ is attributed by David Gale [Ga] to Dana Scott. This too has the Laurent property ([FZ, Theorem 1.10]) and it can be shown (see [Ho1]) that $\deg(f^N)=2,4,8,14,24,40,66,108,\dots=2(F_N-1)$, where F_N denotes the Fibonacci numbers. Hence this map has algebraic entropy $\log\frac{1+\sqrt5}{2}$.

Example 4.7 (Eigentorus)**.** A different instance of positive entropy, much closer to the concerns of this article, is the monomial map $(x,y)\mapsto(y,xy)$ of Example 2.8 associated with the 2-by-2 matrix

$$A=\begin{pmatrix}0&1\\1&1\end{pmatrix}.$$

Here,

$$(x,y)\mapsto(y,xy)\mapsto(xy,xy^2)\mapsto(xy^2,x^2y^3)\mapsto(x^2y^3,x^3y^5)\mapsto(x^3y^5,x^5y^8)\mapsto\dots$$

The exponents are Fibonacci numbers, and the map has algebraic entropy $\frac{1+\sqrt5}{2}$.

The associated projective map $(x\colon y\colon z)\mapsto(yz\colon xy\colon z^2)$ has an "eigentorus" $\{(x\colon y\colon z) : |x|=|y|=|z|\neq 0\}$. One way to think about this eigentorus is to consider the matrix

$$A'=\begin{pmatrix}0&1&1\\1&1&0\\0&0&2\end{pmatrix}$$

obtained from A by adjoining a column of nonnegative integers at the right, in such a fashion that all the row-sums are equal to 2. Let V and V' denote $\mathbb{C}^2$ and $\mathbb{C}^3$, respectively, and give them their standard bases, so that $A\colon V\mapsto V$ and $A'\colon V'\mapsto V'$. The matrix A' has $w=(1,1,1)^T$ as an eigenvector, and we mod out by the eigenspace W; the action of A' on the quotient space V'/W is isomorphic to the action of A on the original 2-dimensional space V. If we now mod out V by the module generated by the two standard unit vectors in $\mathbb{C}^2$ (note: not to be confused with modding out V by the subspace the two vectors span!), corresponding to the fact that $e^{2\pi im+2\pi in}=1=e^0$ for all integers m, n, we get a torus on which A acts as an endomorphism. The same is true in $\mathbb{C}^3$: additively modding out by multiples of $w = (1,1,1)^T$ corresponds to the projective identification (complex dilation) $\sim$ in $\mathbb{C}^3$.

This situation is quite general:

Proposition 4.8. *For any nonsingular matrix A, the action of the monomial map associated with A, restricted to the eigentorus, is isomorphic to the toral endomorphism associated with A.*

Proof. Recall that every monomial map from $\mathbf{CP}^n$ to itself can be written in the form $(x_1 : \ldots : x_{n+1}) \mapsto (p_1(x_1,\ldots,x_{n+1}) : \ldots : p_{m+1}(x_1,\ldots,x_{n+1}))$ where the $m+1$ polynomials $p_1,\ldots,p_{m+1}$ are homogeneous monomials of the same degree (call it d) having no joint common factor. We use the exponents of the $n+1$ variables in the $n+1$ monomials to form an $(n+1)$-by-$(n+1)$ matrix A', and argue as above. □

There is a subtle but important point here, namely, that a monomial map may not be well-defined on all of $\mathbf{CP}^n$, and that even where the monomial map is well-defined, iterates of the map may not be. A brutal way to deal with the problem is to restrict the monomial map to the subset of $\mathbf{CP}^n$ in which all $n+1$ affine coordinate variables are nonzero. A more refined way is to restrict attention to the set $U := \bigcap_{N\geq 1} \operatorname{dom}(f^N)$, the intersection of the domains of the iterated maps $f = f^1, f^2, f^3, \ldots$.

Example 4.9. The projective map $f\colon (x:y:z) \mapsto (yz : xy : z^2)$ from Example 2.8 is not well-defined at $(1:0:0)$ or $(0:1:0)$, and the square of this map is not well-defined at $(1:1:0)$. We could restrict f to the set $\{(x:y:z) : xyz \neq 0\}$, since this restricted map is continuous (and indeed is a homeomorphism), but we could also restrict to the more inclusive set $U = \{(x:y:z) : z \neq 0\}$.

The only truly well-behaved monomial maps are those for which the matrix A is a positive multiple of some permutation matrix. In all other cases, the projective monomial map has singularities:

Example 4.10. Although the affine map $(x, y) \mapsto (x, y^2)$ seems to be nonsingular, it "really" has a singularity at infinity, as we can see when we projectivize it to $(x:y:z) \mapsto (xz : y^2 : z^2)$, which is undefined at $(1:0:0)$.

The typical monomial map has essential singularities; there is no way to extend the partial function to a continuous function defined on all of $\mathbf{CP}^n$.

In this respect, projective monomial maps are somewhat reminiscent of return maps for nonsmooth billiards, which share the property of being undefined on a small portion of the space (corresponding to trajectories in which the ball goes into a corner).

However, unlike the billiards case, in which a seemingly innocuous orbit can be well-defined for millions of steps and then suddenly hit a corner, projective monomial maps have fairly tame sets of singularities, topologically speaking:

Proposition 4.11. *If f is a monomial map from $\mathbf{CP}^n$ to itself, and x is a point in $\mathbf{CP}^n$ for which x, $f(x)$, $f^2(x)$, ..., $f^n(x)$ are all well-defined, then $f^N(x)$ is well-defined for all $N > 2^{n+1}$.*

Proof. Each point in $\mathbf{CP}^n$ can be represented by an $(n+1)$-tuple of 0's and 1's, where a 1 stands for any nonzero complex number. Call this the signature of the point. It is easy to see that the signature of a point determines whether the point is in the domain of f, and in the affirmative case, determines the signature of the image of the point under f. If $f^k(x)$ is well-defined for all $0 \le k \le 2^{n+1}$, then two of the points $f^k(x)$ must have the same signature, so that the sequence of signatures has become periodic, and iteration of the map can be continued indefinitely without fear of hitting the non-point "$(0:0:\cdots:0)$".

The bound 2^{n+1} can actually be replaced by a much smaller bound on the order of n^2, since the way in which the signatures evolve over time correspond to the way in which the entries of the vectors v, Mv, M^2v, … evolve, where M is a nonnegative matrix and v is a nonnegative vector; under this correspondence, 0's correspond to positive entries and 1's correspond to zeroes. For details on the quadratic bound, see [W]. □

Remark 4.12. Proposition 4.11 shows that the set of x in $\mathbf{CP}^n$ for which the infinite forward f-orbit of x is not well-defined is a union of proper subspaces that form a (usually nonpure, i.e., mixed-dimension) complex projective subspace arrangement whose complement U is a dense open subset of $\mathbf{CP}^n$ and is the natural domain on which to investigate the topological dynamics of f.

The dynamics of a monomial map on U can be fairly complicated combinatorially:

Example 4.13. The monomial map $f\colon (x:y:z) \mapsto (xz:xy:z^2)$ associated with the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and its iterates are well-defined on most, but not all, of the complex projective plane $\mathbf{CP}^2$. The points that lie on the projective line $z = 0$ (excepting the point $(0:1:0)$ itself) are mapped by f to the point $(0:1:0)$, which is not in the domain of f. Meanwhile, points on the projective line $y = 0$ are fixed points of f, except for the point $(1:0:0)$ (where the projective line $y = 0$ meets the projective line $z = 0$), which is not in the domain of f. Also, every point on the projective line $x = 0$ is mapped by f to the fixed point $(0:0:1)$.

In the terminology of algebraic geometry, the 1-dimensional subvariety $x = 0$ gets blown down to the 0-dimensional subvariety $x = y = 0$, while the 0-dimensional subvariety $x = z = 0$ gets blown up to the 1-dimensional subvariety $x = 0$ (to see why the latter assertion is true, consider how f acts at points near $(0:1:0)$).

For a discussion of iteration of rational maps that attends to blowing up and blowing down and its implications for degree-growth, see [BK].

5. Topological entropy

5.1. **Choice of entropy.** Recall that Remark 4.12 introduced the set U as the set of points x such that $f^N(x)$ is defined for all $N \ge 1$. Since this dense open subset of $\mathbf{CP}^n$ inherits the angle-metric from the compact space $\mathbf{CP}^n$, we can apply the Bowen–Dinaburg definition of topological entropy [Bo], [Di] by way of spanning or separated sets. But it is desirable to have a more intrinsic way of thinking about the topological dynamics of f. Friedland's approach in such cases (see [Fr1], [Fr2], and [Fr3]) is to compactify the dynamical system inside a countable product of copies of the original space. Specifically, one identifies the point x with the orbit $(x, f(x), f^2(x), \dots)$ in $(\mathbf{CP}^n)^\infty$, and takes the closure of the set of all such orbits; this gives a compact space to which the original Adler–Konheim–McAndrew definition [AKM] can be applied. The results of [HNP] show that these two different ways of defining entropy coincide in the case of monomial maps.

5.2. **Monomials.**

Theorem 5.1. *If A is an n-by-n nonsingular integer matrix, the topological entropy of the monomial map from* $\mathbf{CP}^n$ *to itself associated with A (as in Definition 1.2) is at least the logarithm of the modulus of the product of all the eigenvalues of A outside the unit circle.*

Proof. We use the fact that entropy does not increase when one restricts the dynamical system to an invariant set. Hence, by Proposition 4.8, the topological entropy of the monomial map on $\mathbf{CP}^n$ is at least the topological entropy of the toral endomorphism associated with A. But the topological entropy of a toral endomorphism is the logarithm of the modulus of the product of the eigenvalues that lie outside the unit circle (see [LW] for the history of this result). □

5.3. **Conjectured equality.** We believe that the topological entropy of a monomial map, is exactly equal to the quantity in Theorem 5.1, but we have not found a proof of this.

One way to prove this equality would be to make use of the intermediate dynamical degrees mentioned in Section 3. A theorem of Dinh and Sibony [DS] says that the topological entropy of a map is bounded above by the logarithm of the maximal dynamical degree. If we order the n eigenvalues of A in such a way that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, then it is natural to conjecture that the kth dynamical degree of a monomial map is equal to $|\lambda_1\lambda_2\cdots\lambda_k|$. (This conjecture is true for $k = n$: the product of all the eigenvalues is equal to the determinant of the matrix A, whose absolute value is the degree of the associated monomial map. The conjecture is also true for $k = 1$: this is the content of Theorem 6.2 below.) Note that, as k varies, the maximum value achieved by $|\lambda_1\lambda_2\cdots\lambda_k|$ is equal to the modulus of the product of those eigenvalues that lie outside the unit circle, which is known to equal the topological entropy of the toral endomorphism associated with A. Hence, our conjectural formula for the dynamical degrees of a monomial map, in combination with the theorem of Dinh and Sibony, would imply that the topological entropy of a monomial map is bounded by the topological entropy of the associated toral endomorphism. Since the reverse inequality holds as well (see Subsection 5.1), the desired equality would follow.

6. Algebraic entropy

Recall the formula (2.1) for $D(A)$ that gives the degree of the monomial map associated with the n-by-n matrix A.

Remark 6.1. It is easy to see that composition of affine monomial maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ is isomorphic to multiplication of n-by-n matrices. So computing the degree of the Nth iterate of a monomial map is tantamount to computing $D(A^N)$, and the algebraic entropy of the monomial map is just $\lim_{N\to\infty}(1/N)\log D(A^N)$.

Theorem 6.2. *If A is an n-by-n nonsingular integer matrix, the algebraic entropy of the monomial map from* $\mathbf{CP}^n$ *to itself associated with A is equal to the logarithm of the spectral radius of A.*

Proof. The entries of A^N are $O(r^N)$, where r is the spectral radius of A, so $D(A^N) = O(r^N)$, and the algebraic entropy of the map is at most the logarithm of the spectral radius. To prove equality suppose for the sake of contradiction that $D(A^N) = O(c^N)$ with $1 < c < r$. Replacing c by a larger constant if necessary, we get $D(A^N) < c^N$ for all sufficiently large N. Recalling the formula for $D(\cdot)$, we conclude from this that for large N, every entry of A^N is greater than $-c^N$ and every row-sum of A^N is less than c^N. That is, we now have upper bounds on the row-sums of A^N and on the negatives of the individual entries of A^N; from these, we can derive an upper bound on the entries of A^N. For, since each entry of A^N can be written as the sum of the entries in its row minus the $n-1$ entries in that row other than itself, every entry of A^N is less than nc^N. Hence for every unit vector u (whose components all have modulus less than 1), each component of $A^N u$ has modulus at most $n(nc^N) = n^2 c^N$. Hence the sum of the squares of the entries of $A^N u$ is at most $n(n^2 c^N)^2$, so the norm of $A^N u$ is at most $n^{5/2} c^N$. But when N is large enough, this estimate contradicts the fact that A^N has a unit eigenvector u for which the norm of $A^N u$ is r^N. □

Theorems 5.1 and 6.2 together imply

Corollary 6.3. *For any A with two or more (not necessarily distinct) eigenvalues outside the unit circle, the algebraic entropy of the monomial map associated with A is strictly less than the topological entropy of the map.*

An immediate consequence of Theorem 6.2 is

Corollary 6.4. *The algebraic entropy of the monomial map is equal to the logarithm of an algebraic integer.*

Proof. The spectral radius r of A is an algebraic integer: Let z be a dominant eigenvalue of A, so that $r = |z|$. Since z is an algebraic integer, so is $\overline{z}$, and hence so is $\sqrt{z\overline{z}} = |z| = r$. □

This establishes Conjecture 1.1 for monomial maps.

7. Counterexamples

7.1. **Entropy gap.** Another easy consequence of Theorem 6.2 is

Corollary 7.1. *There exist monomial maps for which the topological entropy is strictly greater than the algebraic entropy.*

Proof. The affine map $(x, y) \mapsto (x^2, y^3)$ has topological entropy $\log 6$ and algebraic entropy $\log 3$. □

7.2. **Inverses.** One might be tempted to conjecture that the algebraic entropy of a birational map is equal to the algebraic entropy of its inverse (since most notions of entropy are preserved by inversion). Toral automorphisms give an easy way to see that this fails in general, because the spectral radius of a matrix that is invertible over $\mathbb{Z}$ is typically not equal to the spectral radius of its inverse:

Example 7.2. Let

$$A := \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with associated monomial map $f\colon (x, y, z) \mapsto (y/x, z/x, x)$. The characteristic polynomial of A is $t^3 + t^2 + t - 1$, whose eigenvalues are approximately 0.54 and $-0.77 \pm 1.12i$. The spectral radius of A is $\sqrt{(-0.77\dots)^2 + (1.12\dots)^2} \approx 1.36$ and the spectral radius of A^{-1} is $1/.54\cdots \approx 1.84$, which is the square of the spectral radius of A. Hence the algebraic entropy of the monomial map f^{-1} is twice the algebraic entropy of f.

7.3. **Degree sequence and linear recurrences.** A more subtle conjecture, due to Bellon and Viallet, is that for any rational map f, the sequence $(\deg(f^N))_{N=1}^{\infty}$ satisfies a linear recurrence with constant coefficients and leading coefficient 1.

If this were true, it would certainly imply that the algebraic entropy of a rational map is always the logarithm of an algebraic integer. However, the map f in Example 7.2 gives a counterexample to this claim:

Proposition 7.3. *For the rational map f in Example 7.2 the sequence $(\deg(f^N))_{N=0}^{\infty} = 1, 2, 3, 4, 6, 9, 12, 17, 25, 33, 45, 65, 85, 112, 159, 215, 262, 365, 524, 627, 833, \ldots$ of degrees does not satisfy any linear recurrence with constant coefficients.*

To see what is going on with this example on an intuitive level, let d_N denote the degree of f^N, and consider the sequence $c_N := -2, \mathbf{2}, 1, -5, \mathbf{6}, 0, -11, \mathbf{17}, -6, -22, \mathbf{45}, -29, -38, \mathbf{112}, -103, -47, \mathbf{262}, -318, 9, 571, -898, \ldots,$ many of whose entries (shown in boldface) agree with the corresponding entries of the degree sequence for f. c_N is the sum of the entries in the last row of A^N minus the sum of the entries on the principal diagonal of A^N. In terms of the notation introduced following Proposition 2.14, $c_N = L_C(A^N)$ for a particular chamber C. It appears empirically that the sequence of matrices $A, A^2, A^3, \ldots$ visits this chamber C infinitely often, so that $c_N = d_N$ for infinitely many values of N. Certainly *some* chamber is visited infinitely often, so for simplicity we will assume that this particular chamber gets visited infinitely often. (The analysis given below does not depend in any essential way on which chamber C is being discussed.)

Proof. The sequence of c_N's satisfies the linear recurrence $c_N = c_{N-3} - c_{N-2} - c_{N-1}$ as a consequence of the Cayley–Hamilton theorem (note that the characteristic polynomial of this recurrence coincides with the characteristic polynomial of the matrix A), so the generating function $\sum_{N=0}^{\infty} c_N x^N$ is the power series expansion of a rational function of x. If the sequence d_N satisfied some linear recurrence with constant coefficients, then the generating function $\sum_{N=0}^{\infty} d_N x^N$ would also be the power series expansion of a rational function of x. It would follow that the generating function $\sum_{N=0}^{\infty}(d_N - c_N)x^N = 3x^0 + 0x^1 + 2x^2 + 9x^3 + 0x^4 + \ldots$ must also be the power series expansions of a rational function of x. It follows from a standard theorem on such expansions due in various versions to Skolem, Mahler, and Lech (see e.g., Exercise 3.a in Chapter 4 of [St]) that the set S consisting of those indices N for which $d_N - c_N = 0$ must be eventually periodic, that is, there must be some union of (one-sided) arithmetic progressions whose symmetric difference with S is finite.

To see that this cannot happen, note that $d_N = c_N$ precisely when several things are simultaneously true of the matrix A^N: For $i = 1, 2, 3$ the j,1-entry of A^N is nonpositive and does not exceed any other entry in its column, and the sum of the entries in the third row of A^N is at least zero and is greater than or equal to both of the other row-sums of A^N. In particular, if $d_N - c_N$ vanishes along some arithmetic progression of values of N, the 1,1 entry of A^N must be nonpositive along some arithmetic progression of values of N.

On the other hand, we can use a basic fact from linear algebra to express the 1,1 entry of A^N as an algebraic function of N. Recall that the set of solutions of a homogeneous linear difference equation with characteristic polynomial $p(t)$ is spanned by the set of sequences of the form $s_N = N^i r^N$ where r is a root of $p(t)$ and i is some nonnegative integer strictly smaller than the multiplicity of r. In particular, there is an exact formula for the 1,1 entry of A^N of the form $c_1\alpha^N + c_2\beta^N + c_3\overline{\beta}^N$, where α is the real root of the characteristic polynomial $t^3 + t^2 + t - 1 = 0$ and $\beta = re^{i\theta}$ and $\overline{\beta} = re^{-i\theta}$ are the complex roots. Since $c_1\alpha^N + c_2\beta^N + c_3\overline{\beta}^N$ is real for all N, c_1 is real and c_2 and c_3 are complex conjugates of one another.

Lemma 7.4. *No power of β is real, i.e., θ is incommensurable with 2π.*

Proof. If there were a positive integer m with β^m real, then α^m, β^m, and $\overline{\beta}^m$ would be the roots of a cubic with rational coefficients possessing a double root $\beta^m = \overline{\beta}^m$; this would imply that α^m and β^m are rational. But α^m, like α itself, is an algebraic integer, so the only way it can be rational is if it is a rational integer; and this cannot be, since it is a nonzero real number with magnitude strictly between 0 and 1. □

Lemma 7.5. *The coefficients c_2 and $c_3 = \overline{c_2}$ are nonzero.*

Proof. If c_2 and c_3 vanish, the 1,1 entry of A^N is always $c_1\alpha^N$. Taking two different values of N for which the 1,1 entry of A^N is an integer, we find that some power of α is rational, and so then is some power of β, contradicting Lemma 7.4. □

Lemma 7.4 implies that for values of N lying in any fixed arithmetic progression, the (complex) values taken on by $(\beta/r)^N$ are dense in the unit circle, and (by Lemma 7.5) the (real) values taken on by $c_2(\beta/r)^N + \overline{c_2}(\overline{\beta}/r)^N$ are dense in some interval centered at 0. In particular, for values of N in that arithmetic progression, $c_1(\alpha/r)^N + c_2(\beta/r)^N + \overline{c_2}(\overline{\beta}/r)^N$ will spend a positive fraction of the time in a ray of the form (ϵ,∞) for some $\epsilon > 0$. This means that the 1,1 entry of A^N, being equal to $c_1\alpha^N + c_2\beta^N + \overline{c_2}\overline{\beta}^N$, will be positive for infinitely many values of N (and hence at least one) in our arithmetic progression. But this contradicts our choice of the arithmetic progression.

Following back the chain of suppositions, we see that we must conclude that the sequence $d_0, d_1, d_2, \ldots$ does not satisfy any linear recurrence with constant coefficients, and our proof is complete. □

More generally, the same reasoning that is given above shows

Proposition 7.6. *Let A be any nonsingular n-by-n matrix whose dominant eigenvalues are a pair of complex numbers $re^{i\theta}$, $re^{-i\theta}$ where θ is incommensurable with 2π. For iterates of the monomial map associated with A, the degree sequence does not satisfy any linear recurrence with constant coefficients.*

Example 7.7. The 2-by-2 matrix

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

associated with the (nonbirational) rational map $(x, y) \mapsto (xy^2, y/x^2)$ has eigenvalues $1 \pm 2i$, and the angle between the lines $y = 2x$ and $y = -2x$ is irrational (i.e., incommensurable with π), so we see that the degree sequence will not satisfy any linear recurrence with constant coefficients.

7.4. **Conjugation.** We have not studied what happens when one starts with a monomial map and conjugates it via a nonmonomial birational map, obtaining (in general) a nonmonomial map. In particular, it seems conceivable that a suitable nonmonomial conjugate of the main counterexample of this paper might be better behaved, in the sense that its degree sequence would satisfy a linear recurrence.

It should be emphasized that the degree sequence associated with a rational map is *not* invariant under birational conjugacy. Conjugating the map f may yield a birational map with a different degree sequence. Indeed, we saw in Example 2.15 that the very first term of the degree sequence, namely the degree of the map itself, may change under birational conjugacy.

7.5. **The price of projectivization.** Jean-Marie Maillard, in private communication, has pointed out that if one works in the affine context and simply studies iterates of the mapping $f\colon (x,y,z) \mapsto (y/x, z/x, x)$ in Example 7.2, one can express the iterates in closed form: $f^N(x,y,z)$ is a triple of monomials, each of which can be written in the form $x^{a_N} y^{b_N} z^{c_N}$ where the sequences $a_1, a_2, \dots$, $b_1, b_2, \dots$, and $c_1, c_2, \dots$ *do* satisfy linear recurrence relations with constant coefficients. (Since there is no projective cancellation going on here, this is just a matter of ordinary linear algebra, in multiplicative disguise.) Maillard suggests through this example that projectivization, although conceptually compelling, may come at a price. In particular, the nonrationality of the degree sequence for iterates of the associated projective map might be viewed as a result of our insistence on working in the projective setting rather than the affine setting.

Note furthermore that projectivization of the affine monomial map does not usually remove singularities, and that projectivization takes a seemingly singularity-free map like $(x,y) \mapsto (x,y^2)$ and tells us that it actually has a singularity at infinity.

8. Piecewise linear maps

Although the main focus of this article has been monomial maps, a general dynamical theory of birational maps would also treat more general maps of the sort considered in Section 4, such as the Scott map $(x,y,z) \mapsto (y,z,(y^2+z^2)/x)$ in Example 4.6. Just as monomial maps are closely associated with linear maps from $\mathbb{R}^n$ to itself (which in turn are closely associated with endomorphisms of the n-torus), certain nonmonomial maps are associated with piecewise linear maps from $\mathbb{R}^n$ to itself.

8.1. **Subtraction-free maps.** We say a map is subtraction-free if each component of the map can be written as a subtraction-free expression in the coordinate variables. E.g., consider the map $f\colon (x,y) \mapsto (x^2+xy+y^2, x^2-xy+y^2)$. Since $x^2-xy+y^2 = (x^3+y^3)/(x+y)$, both components of $f(x,y)$ can be written in terms of x and y using only addition, multiplication, and division. Hence the mapping is subtraction-free. This implies that the iterates of f can also be expressed using only addition, multiplication, and division. The way in which this leads us to consider piecewise linear maps is that the binary operations $(a,b) \to \max(a,b)$, $(a,b) \to a+b$, and $(a,b) \to a-b$, satisfy many of the same properties as the binary operations $(x,y) \to x+y$, $(x,y) \to xy$, and $(x,y) \to x/y$, respectively (with the additive identity element 0 in the former setting corresponding to the multiplicative identity element 1 in the latter setting). More specifically, all of the simplifications that occur when one iterates subtraction-free rational maps are forced to occur when one iterates the associated piecewise linear maps. So, for example, the cancellations that permit the rational map $(x,y) \mapsto (y,(y+1)/x)$ to be of order 5 force the piecewise linear map $(a,b) \mapsto (b, \max(b,0)-a)$ to be of order 5 as well.

The operation on subtraction-free expressions that replaces multiplication by addition, division by subtraction, and addition by max, or min, has attracted a good deal of attention lately; it is known as "tropicalization", and a good introduction to the topic is [SS].

Example 8.1. It is interesting to compare $(x,y) \mapsto (y,(y^2+1)/x)$ from Example 4.2 with $(a,b) \mapsto (b, \max(2b,0)-a)$. Iteration of the former map gives rise to the sequence of rational functions x, y, $\dfrac{y^2+1}{x}$, $\dfrac{y^4+x^2+2y^2+1}{x^2y}$, $\dfrac{y^6+x^4+2x^2y^2+3y^4+2x^2+3y^2+1}{x^3y^2}$, …, while iteration of the latter map gives rise to the sequence of piecewise linear functions

$$\begin{aligned}&\max(-1a-0b,-1a+2b,-1a-0b),\\&\max(\;\;0a-1b,-2a+3b,-2a-1b),\\&\max(\;\;1a-2b,-3a+4b,-3a-2b),\\&\max(\;\;2a-3b,-4a+5b,-4a-3b),\end{aligned}$$

etc. (Note that the first of these piecewise linear functions can be written more simply as $\max(-a,-a+2b)$, but expressing it in a more redundant fashion brings out the general pattern.)

Example 8.2. It is even more interesting to consider the piecewise linear analogue of the Scott map $(x,y,z)\mapsto(y,z,(y^2+z^2)/x)$ from Example 4.6, which is $(a,b,c)\mapsto(b,c,\max(2b,2c)-a)$. Iteration of the latter map gives rise to the sequence of piecewise linear functions

$$\begin{aligned}&\max(\;-1a\;\;+2b-0c,\;\;-1a\;\;-0b\;\;+2c,-1a\;\;+0b+2c,-1a\;\;+2b+0c),\\&\max(\;-2a\;\;+3b-0c,\;\;-2a\;\;-1b\;\;+4c,\;\;\;0a\;\;-1b+2c,-2a\;\;+3b-0c),\\&\max(\;-4a\;\;+6b-1c,\;\;-4a\;\;-2b\;\;+7c,\;\;\;0a\;\;-2b+3c,-2a\;\;+4b\;\;-c),\\&\max(\;-7a+10b-2c,\;\;-7a\;\;-4b+12c,\;\;\;1a\;\;-4b+4c,-3a\;\;+6b-2c),\\&\max(-12a+17b-4c,-12a\;\;-7b+20c,\;\;\;2a\;\;-7b+6c,-4a\;\;+9b-4c),\\&\max(-20a+28b-7c,-20a-12b+33c,\;\;\;4a-12b+9c,-6a+14b-7c),\end{aligned}$$

etc., in which the coefficients can be expressed in terms of Fibonacci numbers. The Lipschitz constants of these maps grow exponentially, with asymptotic growth rate given by the golden ratio.

8.2. **Lyapunov growth.** More generally, when one compares a subtraction-free rational recurrence with its piecewise linear analogue, one often finds that the growth rate for the Lipschitz constants of iterates of the piecewise linear map (which one can view as a kind of global Lyapunov exponent) is equal to the growth rate for the degrees of iterates of the rational map. In fact, every cancellation that occurs when one iterates the rational map also occurs when one iterates the piecewise linear map, so the algebraic entropy of the former is an upper bound on the logarithm of the global Lyapunov exponent of the latter.

Remark 8.3. Purists may note that we are modifying the usual notion of Lyapunov exponent in several respects. First, we are re-ordering quantifiers. Ordinarily one looks at the forward orbit of a specific point x, and sees how the maps f^N expand neighborhoods of x with N going to infinity, and only after defining this limit does one let x vary over the space as a whole; here we are taking individual values of N and for each such N we ask for the largest expansion that f^N can cause on the whole space. Another difference is that our piecewise linear maps are not differentiable, so we are using Lipschitz constants as a stand-in for derivatives.

8.3. **PL maps and PL recurrences.** It may seem that we have wandered a bit from the main themes of this article, but the reader may recall that piecewise linear maps entered the article fairly early on, via the formula (2.1).

Example 8.4. The affine Scott map $(x,y,z)\mapsto(y,z,(y^2+z^2)/x)$ from Example 4.6 (and Example 8.2) gives rise to a sequence of Laurent polynomials whose denominators are $x^1y^0z^0$, $x^2y^1z^0$, $x^4y^2z^1$, $x^7y^4z^2$, $x^{12}y^7z^4$, $x^{20}y^{12}z^7$, … where the exponent-sequence 0, 1, 2, 4, 7, 12, 20, ... is associated with iteration of the piecewise linear map $(a,b,c) \mapsto (b,c,\max(2b,2c)-a)$ we associated with the Scott map in Example 8.2.

Example 8.5. Consider the affine monomial map $f\colon (x,y,z) \mapsto (y/x, z/x, x)$ of Example 7.2 discussed in Subsection 7.5. If we write $f^N(x,y,z)$ as

$$(p_N^1(x,y,z)/q_N^1(x,y,z), p_N^2(x,y,z)/q_N^2(x,y,z), p_N^3(x,y,z)/q_N^3(x,y,z))$$

where (for $1 \le i \le 3$) p_N^i and q_N^i are monomials with no common factor, then we can write each sequence $p_1^i, p_2^i, p_3^i, \ldots$ or $q_1^i, q_2^i, q_3^i, \ldots$ in the form $x^{a_1}y^{b_1}z^{c_1}$, $x^{a_2}y^{b_2}z^{c_2}$, $x^{a_3}y^{b_3}z^{c_3}$, ... where each of the sequences $a_1, a_2, a_3, \ldots$, $b_1, b_2, b_3, \ldots$, and $c_1, c_2, c_3, \ldots$ satisfies a linear recurrence. Indeed, it is possible that the degree sequence for iterates of the projective monomial map $(w\colon x\colon y\colon z) \mapsto (wx\colon wy\colon wz\colon x^2)$ (the projectivization of f) satisfies a *piecewise* linear recurrence, but we have not explored this. (For a simple example of an integer sequence that satisfies a piecewise linear recurrence but does not appear to satisfy any linear recurrence with constant coefficients, consider the sequence $1,1,-1,-1,-3,1,3,9,7,3,-11,-11,-17,11,33,67,45,1,\ldots$ satisfying the recurrence $a_n = \max(a_{n-1}, a_{n-2}) - 2a_{n-3}$.)

8.4. **PL projectivization.** As a final note, we mention that projectivization has an analogue in the piecewise linear context, namely, modding out (additively) by multiples of $(1,1,1)$.

Example 8.6. Consider once again the map $(a,b,c) \mapsto (b,c,\max(2b,2c)-a)$ from Examples 8.2 and 8.4. It sends $(a',b',c') = (a,b,c)+(d,d,d)$, to $(b,c,\max(2b,2c)-a)+(d,d,d)$, that is, it commutes with adding constant multiples of $(1,1,1)$, so we can consider a quotient action that acts on equivalence classes of triples, where two triples are equivalent if their difference is a multiple of $(1,1,1)$.

This quotient construction applies whenever our piecewise linear map is "homogeneous", in the sense that there exists a constant m such that each component of the piecewise linear map is a max of linear functions, all of which have coefficients adding up to m. (In the example we just considered, $m = 1$.)

9. Comments and open questions

We suggest that in some respects, the logarithm of the maximal dynamical degree behaves in a fashion more analogous with other kinds of entropy than Bellon and Viallet's notion of algebraic entropy does. (Some of our e-mail correspondents have taken this point of view as well.) In the case of a monomial map associated with a nonsingular integer matrix A, we have already shown that algebraic entropy as defined by Bellon and Viallet is the spectral radius of A, whereas the logarithm of the maximal dynamical degree of the map stands a decent chance of being equal to the topological entropy of the toral endomorphism associated with A. Furthermore, Tien-Cuong Dinh has pointed out to us in private correspondence that if f is any birational map from projective n-space to itself, the kth dynamical degree of f is equal to the $n-k$th dynamical degree of f^{-1} (as a trivial consequence of the equality between $\int (f^N)^*(\omega^k)\wedge\omega^{n-k}$ and $\int \omega^k \wedge (f^{-N})^*\omega^{n-k}$ obtained by a coordinate change), from which it easily follows that the logarithm of the maximal dynamical degree of f^{-1} equals the logarithm of the maximal dynamical degree of f.

Question 9.1. Is the algebraic entropy of a monomial map always equal to the topological entropy of the associated toral endomorphism?

Question 9.2. Is the algebraic entropy of a map always bounded above by its topological entropy?

We have seen that this is true for monomial maps. The discussion in Subsection 5.3 is pertinent. Also see [Ng].

A different sort of question about inequalities is:

Question 9.3. Is algebraic entropy nonincreasing under factor maps?

That is, if we have birational maps $f\colon \mathbf{CP}^n \mapsto \mathbf{CP}^n$ and $g\colon \mathbf{CP}^m \mapsto \mathbf{CP}^m$, and a rational map $\phi\colon \mathbf{CP}^n \mapsto \mathbf{CP}^m$ satisfying

$$\phi \circ f = g \circ \phi,$$

must the algebraic entropy of g be less than or equal to the algebraic entropy of f?

To avoid trivial counterexamples, we should insist that the map be dominant (i.e., that its image is Zariski-dense in $\mathbf{CP}^m$); here, this is equivalent to assuming $n \geq m$.

Of continuing importance is the Conjecture 1.1 of Bellon and Viallet:

Question 9.4. Is the algebraic entropy of a rational map always the logarithm of an algebraic integer?

One might also try to clarify the situation for the case in which algebraic entropy vanishes.

Question 9.5. Can the degree sequence of a rational map be subexponential but super-polynomial?

Question 9.6. If the degree sequence of a rational map is bounded above by a polynomial, must it grow like N^k for some nonnegative integer k, or can it exhibit intermediate asymptotic behavior, such as $\sqrt{N}$?

Even though monomial maps provide counterexamples to Bellon and Viallet's conjecture about degree sequences, it surely cannot be a mere coincidence that so many of the examples studied by Bellon and Viallet and others have the property that the degree sequences satisfy recurrence relations with constant coefficients. So one might inquire whether we can rescue Bellon and Viallet's conjecture on degree sequences by adding extra hypotheses. One such possible extra hypothesis is suggested by the fact (pointed out to us by Viallet) that many of the birational mappings studied by Bellon and Viallet can be written as compositions of involutions.

Question 9.7. If a rational map is a composition of involutions, must its degree sequence satisfy a linear recurrence with constant coefficients?

It may be worth mentioning that, under the hypothesis of Question 9.7, the rational map is birationally conjugate to its inverse, so the two maps have the same algebraic entropy.

Question 9.8. Must the degree sequence of a rational map satisfy a piecewise linear recurrence with constant coefficients?

Question 9.9. Is there a simple formula for the intermediate dynamical degrees of monomial maps, generalizing Proposition 2.14?

Intermediate dynamical degrees (first defined in [RS]), although conceptually quite natural, have proved to be difficult to compute in all but the simplest of cases; monomial maps constitute a setting in which one might hope to do computations and prove nontrivial results. It is natural to conjecture that the kth dynamical degree of a monomial map is equal to $|\lambda_1\lambda_2\cdots\lambda_k|$, where $\lambda_1,\lambda_2,\ldots$ are the eigenvalues of the associated matrix, ordered so that $|\lambda_1| \geq |\lambda_2| \geq \ldots$. As was remarked in Subsection 5.3, a proof of this conjecture for all k would yield an affirmative answer to Question 9.1.

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