—TEST—

This is a closed book exam. Put all your answers in the blue exam book. When you see [T/F] (short for “True / False”) please decide whether that statement is true or false and accordingly prove the statement carefully or give an explicit counterexample. Choose five of the problems below to work on. If you hand in more than five solutions I will grade the first five of those (in numerical order). Each problem is worth ten points. You are allowed to use without proof that

\[
\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \begin{cases} 
-1 & -\pi < x < 0 \\
1 & 0 < x < \pi \\
0 & x = 0, \pm \pi 
\end{cases}
\]

1. [T/F] If \( f \) is sectionally continuous on \([-\pi, \pi]\), \( f(-\pi) = f(\pi) \), and \( f' \) is sectionally continuous with jump discontinuities then the Fourier series of \( f \) converges uniformly to \( f \).

If you decide the statement is true please give a reason, i.e., at least a heuristic argument. Otherwise give a reason why the statement fails, ideally a counterexample.

2. Suppose \( f: [a, b] \times [c, d] \to \mathbb{R}, (x, t) \mapsto f(x, t) \) is continuous and \( \partial f/\partial t \) exists and is continuous on \([a, b] \times [c, d]\). Let \( u, v: [c, d] \to [a, b] \) be continuously differentiable and \( F(t) := \int_{u(t)}^{v(t)} f(x, t) \, dx \). Prove that \( F \) is differentiable and

\[
F'(t) = f(v(t), t)v'(t) - f(u(t), t)u'(t) + \int_{u(t)}^{v(t)} \frac{\partial f}{\partial t}(x, t) \, dx.
\]

3. a. Define “complete orthonormal system” (i.e., define both “orthonormal system” and “complete”).

b. Suppose \( \{\varphi_1, \varphi_2, \ldots\} \) is a complete orthonormal system. Prove that \( \{\varphi_2, \varphi_3, \ldots\} \) is not a complete orthonormal system.

4. Find the Fourier series of \( f(x) := \begin{cases} 
-\pi/4 & -\pi < x < 0 \\
\pi/4 & 0 < x < \pi \\
0 & x = 0, \pm \pi
\end{cases} \)

5. Suppose \( V \) is an inner product space and \( x_n \to x \) as \( n \to \infty \). [T/F] If \( y \in V \) then \( \langle x_n, y \rangle \to \langle x, y \rangle \).

6. Consider the set of step functions \( [0, 1] \to \mathbb{R} \), i.e., the functions \( f \) for which there is a finite partition \( 0 = x_0 < x_1 < \cdots < x_n = 1 \) (depending on \( f \), of course) such that \( f \) is constant on \([x_{i-1}, x_i]\). Prove that this is a linear space (with the usual pointwise addition and scalar multiplication).

7. Suppose \( f: [0, 1] \to \mathbb{R} \) is a bounded Riemann integrable function. [T/F] For every \( \epsilon > 0 \) there is a step function \( s: [0, 1] \to \mathbb{R} \) such that \( |\int f(x) - s(x) \, dx| < \epsilon \).