When you see [I/E] please decide whether that statement is true or false and accordingly prove the statement carefully or give an explicit counterexample. This exam is due Friday, May 4 at 4:00 p.m. You may use the book and any class notes, but you cannot collaborate. Choose ten problems to work on. If you hand in more I will grade the first ten. Each problem is worth ten points. Give detailed (but not verbose) proofs.

1. Let $C \subset \mathbb{R}^n$ be closed such that $x \in C \implies \alpha x \in C$ for all $\alpha \geq 0$. [I/E] If $f: C \rightarrow \mathbb{R}^m$ is continuous and $f(\alpha x) = \alpha f(x)$ for $x \in C$, $\alpha \geq 0$ then there is an $M \in \mathbb{R}$ such that $\|f(x)\| \leq M \|x\|$ for all $x \in C$.

2. Consider $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x,y) := \begin{cases} x + \frac{\sin((x^2 + y^2)^{-1/2})}{x^2 + y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$.

For each $(x,y) \in \mathbb{R}^2$ decide whether $f$ is differentiable at that point, and if so, find the derivative at that point.

3. Define $l(c) := \sup \{ \sum_{i=1}^{n} \|c(t_i) - c(t_{i-1})\| \mid 0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1 \}$ for any continuous curve $c: [0,1] \rightarrow \mathbb{R}^n$. [I/E] If $c$ is differentiable then $l(c) = \int_0^1 \|c'(t)\| \, dt$.

4. Suppose $O \subset \mathbb{R}^2$ is open and path-connected, $f: O \rightarrow O$ differentiable with $\|Df(x)\| \leq 1/2$ for all $x \in O$. [I/E] $f$ is a contraction on $O$.

5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be differentiable, $0 \leq f'(x) \leq f(x)$ for all $x \in \mathbb{R}$. Show that if $f(x) = 0$ for some $x \in \mathbb{R}$ then $f(x) = 0$ for all $x \in \mathbb{R}$.

6. Suppose $f: \mathbb{R} \rightarrow \mathbb{R}^2$, $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $g \circ f$ differentiable, $Dg(f(0)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

[I/E] $f$ is differentiable at 0.

7. Show that the Implicit Function Theorem implies the Inverse Function Theorem.

8. Prove that the fixed point of a contraction depends smoothly on the contraction:

Suppose $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is $C^r$ and there exists a $\lambda < 1$ with $d(f(x,y), f(x',y)) \leq \lambda d(x,x')$ for $x,x' \in X$. Then for every $y \in Y$ there is a unique fixed point $g(y)$ of $x \mapsto f(x,y)$. Prove that $g$ is $C^r$. (Hint: Use the Implicit Function Theorem.)

9. Prove carefully that the ternary Cantor set has zero volume (“has volume” & “= 0”).

10. Prove $f(x) := \begin{cases} 0 & 0 \leq x < 1/2 \\ 1 - x & 1/2 \leq x \leq 1 \end{cases}$ is integrable on $[0,1]$ using Definition 8.1.1.

11. For which $p \in \mathbb{R}$ does $\int_1^\infty \sin((\log x)^p) / x \, dx$ exist?

12. Compute $\int_0^\pi \int_y^\infty (\sin x) / x \, dx \, dy$.

13. [I/E] If $f, f_n: [0,1] \rightarrow \mathbb{R}$ continuous, $f_n \rightarrow f$ pointwise then $f_n \rightarrow f$ in the mean.

14. Suppose $p: [0,1] \rightarrow \mathbb{R}$ is continuous. Give necessary and sufficient conditions on $p$ under which $\langle f, g \rangle := \int_0^1 f(x) \overline{g(x)} p(x) \, dx$ defines an inner product on the space of complex-valued continuous functions on $[0,1]$.

15. Give the Fourier series of $f(x) := \begin{cases} 1 & -\pi < x < -\pi/2 \text{ or } 0 < x < \pi/2 \\ -1 & -\pi/2 < x < 0 \text{ or } \pi/2 < x < \pi \end{cases}$ on $[-\pi, \pi]$ (use sin and cos, not exponentials).