1. (15 points) Consider the differential equation \( x \frac{dx}{dt} = -t^2 \).

a. Is this differential equation linear? Explain!

Solution: No, because of the \( xx' \)-term—the differential equation can not be brought into the form \( a_1(t)x' + a_0(t)x = E(t) \).

b. Find the solution for which \( x(0) = -5 \).

Solution: Separation of variables: \( x^2 / 2 = \int x \, dx = - \int t^2 \, dt = -t^3 / 3 + C \) or with different \( C \).

\[ x^2 = -2t^3 / 3 + C. \]

Insert \( t = 0, x = -5 \) to get \( 25 = (-5)^2 = C \), so \( x = -\sqrt{25 - 2t^3 / 3} \).

2. (5 points) Show that the functions \( t^3 \) and \( t^4 \) are solutions of \( t^2 x'' - 6tx' + 12x = 0 \).

Solution: Plug them in:

\[ t^2 \cdot 3 \cdot 2t - 6t \cdot 3t^2 + 12t^3 = 0 \quad \checkmark \]

\[ t^2 \cdot 4 \cdot 3t^2 - 6t \cdot 4t^3 + 12t^4 = 0 \quad \checkmark. \]

3. (5 points, no partial credit) Find all solutions of \( (tD^2 - D)x = 0 \) that are of the form \( t^\alpha \).

Solution: Plug in \( t^\alpha \) to get \( (\alpha(\alpha - 1) - \alpha)t^{\alpha - 1} = 0 \) for all \( t \), hence \( \alpha = 0 \) and \( \alpha = 2 \) are the only solutions, so \( 1 \) and \( t^2 \) are (all) solutions of the differential equation of the form \( t^\alpha \).

\[ \begin{pmatrix} 0 & 1 & 6 & 9 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \]

4. (5 points, no partial credit) Evaluate \( \det \left( \begin{array}{cccc} 0 & 1 & 6 & 9 \\ 0 & 2 & 7 & 0 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 4 \end{array} \right) \).

Solution: \( 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120. \)

5. (10 points) A savings account pays 3% interest per year, compounded continuously. In addition, the income from another investment is credited to the account continuously, at the rate of $700 per year. Set up a differential equation to model this account.

Solution: \( x' = 0.03x + 700. \)

6. (5 points, no partial credit) Find the general solution of \( (D - 1)^2(D + 1)x = 0 \).

Solution: \( c_1 e^t + c_2 te^t + c_3 e^{-t}. \) \[ \text{[This should take 15 seconds.]} \]

7. (7 points, no partial credit) Find the general solution of \( 3(D^2 + D + 2)^2 x = 0 \).

Solution: \( 3e^{-t^2/2}((c_1 + c_2 t) \cos \frac{\sqrt{7}t}{2} + (c_3 + c_4 t) \sin \frac{\sqrt{7}t}{2}). \) \[ \text{[Should take 1 minute.]} \]
8. (8 points, no partial credit) Make a simplified guess for a particular solution of

\[(D - 1)(D + 2)^3(D + 2)x = t^2e^t + e^{-t} \sin 3t + t.\]

Do not evaluate the constants.

**Solution:** \(c_1te^t + c_2t^2e^t + c_3t^3e^t + c_4e^{-t} \sin 3t + c_5e^{-t} \cos 3t + c_6 + c_7t.\) [Should take 1 min.]

9. (15 points) Find (and simplify where possible) the general solution of \(x'' - 2x' + x = e^t/t^2.\) (Check all your intermediate answers carefully; no credit for work based on wrong prior steps.)

**Solution:** General solution of the associated homogeneous differential equation \(x'' - 2x' + x = 0:\)

\(H(t) = c_1e^t + c_2te^t;\) variation of parameters gives the solution \(p(t) = c_1(t)e^t + c_2(t)te^t\) with

\[
\begin{align*}
\frac{d}{dt}(c_1(t)e^t + c_2(t)te^t) & = e^t, \\
\frac{d}{dt}(c_1(t)e^t + c_2(t)(t+1)e^t) & = e^t/t^2, \\
\frac{d}{dt}(c_1(t)e^t + c_2(t)(t+2)e^t) & = 1/t^2,
\end{align*}
\]

so \(c_2'(t) = 1/t^2, c_1'(t) = -1/t,\) hence \(c_1(t) = -1/t, c_2(t) = -\ln t,\) and \(p(t) = -e^t \ln t - \frac{1}{t}te^t.\) Therefore \(x(t) = c_1e^t + c_2te^t - e^t \ln t.\)

10. (10 points)
   a. Compute the Wronskian of \(h_1(t) = te^t\) and \(h_2(t) = t^2e^t\) at \(t = 1.\)

**Solution:** \(W[te^t, t^2e^t] = \left| \begin{array}{cc} te^t & t^2e^t \\ (t+1)e^t & (t^2+2t)e^t \end{array} \right| = \left| \begin{array}{cc} te^t & t^2e^t \\ e^t & 2te^t \end{array} \right| = e^2\) for \(t = 1.\)

b. Are these 2 functions linearly independent?

**Solution:** Yes, because \(e^2 \neq 0.\)

11. (10 points) Are the functions \(t^5, |t|^5\) linearly independent on \((-\infty, \infty)\)? Justify your conclusion.

**Solution:** They are: If \(c_1t^5 + c_2|t|^5 = 0\) for all \(t\) then for \(t = 1\) we get \(c_1 + c_2 = 0\) and for \(t = -1\) we get \(-c_1 + c_2 = 0.\) Adding and subtracting these two equations gives \(c_1 = c_2 = 0.\)

12. (5 points) Suppose \(f(t)\) is continuous. Solve the initial-value problem \(x' + f(t)x = 0,\) \(x(1) = 0.\) (Hint: Think before applying standard techniques.)

**Solution:** By inspection, \(x(t) = 0\) (for all \(t\)) is a solution that satisfies the initial condition.

One could get this by separation of variables (but why would you?): Rewrite as \(\frac{dx}{dt} = -f(t)x\) and separate variables to get \(\frac{dx}{x} = -f(t)dt\) (if \(x \neq 0\)). Integrating (and omitting absolute values), we get \(\ln x = -\int f(t) dt + C,\) so \(x = e^C e^{-\int f(t) dt} = Ae^{-\int f(t) dt}\) for some constant \(A\) (which used to be positive but in retrospect does not have to be). To satisfy the initial condition, we must take \(A = 0,\) which gives \(x(t) = 0\) for all \(t.\)