1. (10 points) Show that \( x(t) = t + 4 \) is a solution of the differential equation \( \sin(t)D^3x + 4Dx - x = -t \).

**Solution:** \( \sin(t)D^3x + 4Dx - x = 0 + 4 \cdot 1 - (t + 4) = -t \).

2. (10 points) Write \( \sin(t)D^3x + 4Dx - x = -t \) as a system of differential equations.

**Solution:** Write \( x_1 = x, x_2 = x', x_3 = x'' \) to get

\[
\begin{align*}
x_1' &= x_2 \\
x_2' &= x_3 \\
x_3' &= \frac{1}{\sin t} x_1 - 4 \frac{1}{\sin t} x_2 - \frac{t}{\sin t}
\end{align*}
\]

or

\[
D\vec{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sin t} & -\frac{4}{\sin t} & 0 \end{pmatrix} \vec{x} - \begin{pmatrix} 0 \\ 0 \\ \frac{t}{\sin t} \end{pmatrix}
\]

3. (10 points) Find all solutions (in vector form) to the equation \( D\vec{x} = \begin{pmatrix} 5 & 3 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} \vec{x} \).

**Solution:** \( \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \) are eigenvectors for 1, 7, respectively, and \( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \), \( \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \) are generalized eigenvectors for the double eigenvalue 2 (obtained by solving \( 0 = \det \begin{pmatrix} 5 - \lambda & 3 & 0 & 0 \\ -3 & -1 - \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} \)).

\[
\vec{x}(t) = c_1 e^t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3t \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right] + c_4 e^{2t} \left[ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3t \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right].
\]
4. (10 points) Find all solutions to the equation \( D\vec{x} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ e^t \end{pmatrix} \)

Solution: By inspection, the general solution of the associated homogeneous differential equation is \( \vec{h}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \) (Without inspection, do the usual: find the eigenvalues 3 (with eigenvector \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)) and 5 (with eigenvector \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)) to get this general solution.)

Variation of parameters: \( \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 1 \\ e^t \end{pmatrix} \) gives \( c_1 = -e^{-3t}/3, \quad c_2 = -e^{-4t}/4 \) and \( \vec{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{3} e^t/4. \)

5. (10 points) Use the Laplace transform to solve the initial value problem \( D^2x + Dx - 2x = 2 \) with \( x(0) = -1 \) and \( x'(0) = 0. \) No credit by any other method.

Solution: \( \mathcal{L}[x] + s + 1 = \frac{2}{s}, \) so \( \mathcal{L}[x] = \frac{-s^2 - s + 2}{s(s^2 + s - 2)} = -\frac{1}{s} \) and \( x(t) = -1 \) for all \( t. \)

6. (10 points) Find all solutions to the differential equation \((D - 1)^2 + 1)(D + 3)Dx = 12. \)

Solution: The general solution of the associated homogeneous differential equation is \( h(t) = c_1 e^t \cos t + c_2 e^t \sin t + c_3 e^{-3t} + c_4; \) a simplified guess for a particular solution is \( kt; \) plug this in to get \( 6k = 12, \) so the general solution is \( x(t) = c_1 e^t \cos t + c_2 e^t \sin t + c_3 e^{-3t} + c_4 t + 2t. \)

7. (20 points) Consider the system \( \frac{dx}{dt} = -x - y^2 \quad \frac{dy}{dt} = y(2 - x). \)

a. Is the function \( E(x, y) = x^2 - y^2 \) a constant of motion?

Solution: \( 2x(-x - y^2) - 2y \cdot y(2 - x) = -2x^2 - 4y^2 < 0 \) except when \( (x, y) = (0, 0), \) so no.

b. Is the function \( E(x, y) = x^2 - y^2 \) a Lyapunov function?

Solution: Yes, by the computation in a, it (strictly!) decreases along nonconstant integral curves.

c. Find all equilibria of this system.

Solution: \( 0 = x' = -x - y^2 \) and \( 0 = y' = y(2 + y^2) \) give \( y = 0, \quad x = -y^2 = 0. \) So \((0, 0)\) is it.

d. Find the linearization at each equilibrium and decide whether its phase portrait matches any of those at the end of the examination sheet; if so, identify which of these pictures it matches.

Solution: Dropping higher-order terms, the linearization at \((0, 0)\) is \( \frac{dx}{dt} = -x, \quad \frac{dy}{dt} = 2y. \) (Or plug \((x, y) = (0, 0)\) into \( A(x, y) = \begin{pmatrix} \frac{\partial}{\partial x} (-x - y^2) \\ \frac{\partial}{\partial x} (y(2 - x)) \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \)) has eigenvalues \(-1\) (eigenvector \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)) and \(2\) (eigenvector \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)); this matches \( V. \)

e. For each equilibrium decide whether the Hartman–Grobman Theorem applies.

Solution: Yes, the eigenvalues of the linearization matrix have real parts \(-1\) and \(2; \) both nonzero.

f. Classify each equilibrium as stable or unstable.

Solution: Unstable by parts \( d \) and \( e. \)

g. Classify each equilibrium as attractor, repeller or neither.

Solution: Neither, by parts \( d \) and \( e. \)

h. Are there closed integral curves for this system?
Solution: No, for several reasons: • A closed integral curve must go around an equilibrium, and the only equilibrium here is a saddle, which makes this impossible. • Since \( y' = y(2 - x) \), solutions can not cross the \( x \)-axis, so a closed integral curve would have to lie in the upper half-plane or in the lower half-plane, neither of which contains an equilibrium around which to go. • When \( x = 0 \) then \( x' = -y^2 \leq 0 \), so integral curves cannot cross from the left half-plane to the right half-plane, so a closed integral curve would have to lie in the left half-plane or in the right half-plane, neither of which contains an equilibrium around which to go. • There is a Lyapunov function (which strictly decreases along nonconstant solutions; see a., b.!).

8. (10 points) Consider the system \( \frac{dx}{dt} = x^2 - y^2 \quad \frac{dy}{dt} = yx - y. \)

a. Find all equilibrium points.

Solution: \( 0 = \frac{dy}{dt} = yx - y = y(x - 1) \) implies \( y = 0 \) or \( x = 1 \). If \( 0 = \frac{dx}{dt} = x^2 - y^2 \), then \( x^2 = y^2 \), so \( y = 0 \) implies \( x = 0 \) and \( x = 1 \) implies \( y = \pm 1 \). So they are \((0, 0)\), \((1, 1)\) and \((1, -1)\).

b. For each equilibrium decide whether the phase portrait of the linearization matches any of those at the end of the examination sheet; if so, identify which of these pictures it matches.

Solution: The linearization matrix at \((x, y)\) is \( A(x, y) = \begin{pmatrix} 2x & -2y \\ y & x - 1 \end{pmatrix} \), so

\[ A_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \] this brings about the general solution \( \begin{pmatrix} c_1 \\ c_2 e^{-t} \end{pmatrix} \), as shown in IV.

\[ A_{(1,\pm1)} = \begin{pmatrix} 2 & \mp 2 \\ \pm 1 & 0 \end{pmatrix} \] has eigenvalues \( 1 \pm i \), hence outward spirals. Take \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) as a test point to see that at \((1, 1)\) they go counterclockwise, as in III. and at \((1, -1)\) they go clockwise, as in II.

c. For each equilibrium decide whether the Hartman–Grobman Theorem applies.

Solution: \((0, 0)\): No because 0 is an eigenvalue. \((1, 1)\): Yes—all eigenvalues have real part \( 1 \neq 0 \).

9. (10 points)

a. Find a recursion formula for the coefficients in the power series (centered at \( 0 \)) for the solution of \( D^2x - tDx + x = 0 \) with \( x(0) = 0, \ x'(0) = 1 \).

Solution: Plug in \( x(t) = \sum_{k=0}^{\infty} b_k t^k \) to get \( \sum_{k=0}^{\infty} (k + 2)(k + 1)b_{k+2}t^k - b_k \cdot kt^k + b_k t^k = 0 \) and \( b_{k+2} = \frac{k - 1}{(k + 2)(k + 1)} b_k \).

b. Find the power series.

Solution: The initial values give \( b_0 = 0, b_1 = 1 \), so the recursion gives \( b_k = 0 \) for all even \( k \), \( b_3 = 0 \), and therefore \( b_k = 0 \) for all odd \( k \) thereafter. So \( x(t) = \sum_{k=0}^{\infty} b_k t^k = t \). (Which checks!)

Phase portraits for matching up:

I: ![Phase portrait I]
II: ![Phase portrait II]
III: ![Phase portrait III]
IV: ![Phase portrait IV]
V: ![Phase portrait V]