On the dynamics of chaotic spiking-bursting transition in the Hindmarsh–Rose neuron

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The paper considers the neuron model of Hindmarsh–Rose and studies in detail the system dynamics which controls the transition between the spiking and bursting regimes. In particular, such a passage occurs in a chaotic region and different explanations have been given in the literature to represent the process, generally based on a slow-fast decomposition of the neuron model. This paper proposes a novel view of the chaotic spiking-bursting transition exploiting the whole system dynamics and putting in evidence the essential role played in the phenomenon by the manifolds of the equilibrium point. An analytical approximation is developed for the related crucial elements and a subsequent numerical analysis signifies the properness of the suggested conjecture. © 2009 American Institute of Physics. [DOI: 10.1063/1.3156650]

Many biological neuron models have been developed in the last decades for an accurate description and prediction of biological phenomena. From the basic contribution of Hodgkin and Huxley, the simplified model of Hindmarsh and Rose has turned out to be quite accurate in capturing the features of electrical data from experiments. This model is a three-dimensional oscillator, particularly aimed to study the spiking-bursting behavior of the neuron membrane potential. Indeed, these operating modes and their relationships are key questions in understanding the different mechanisms which regulate the neuronal coding of information in many biological processes. The Hindmarsh–Rose model exhibits the spiking-bursting passage in a chaotic region and such transition has been explained in the past by applying a slow-fast decomposition of the related equations. This paper formulates a novel conjecture to detect the phenomenon. From the complete model, by considering the equilibrium point and its manifolds, a kind of trajectory “separatrix” is analytically derived in the state space for the occurrence of the transition. The method is developed for a general class of polynomial systems, while a numerical analysis, which appears to fully confirm the conjecture, is performed for the specific case in study.

I. INTRODUCTION

The study of the neural activity and its response mechanism to external stimuli has produced in the literature a variety of neuronal models. In this respect it is to underline the importance of the early work of Hodgkin and Huxley, whose model surely represents a milestone. However, since such a model turns out to be quite complex, several authors have attempted to provide simpler approximations. These models are designed to guarantee a qualitative description of some neuronal phenomena in order to derive useful insights about them. Therefore, it is natural that the first ones were low-dimensional models, namely, second order systems such as the FitzHugh–Nagumo and Morris–Lecar neurons.

In this work we refer to the Hindmarsh–Rose neuronal model. Its first version is dated in 1982 (Ref. 7) and again it is a second order form. Despite the genesis of several simplified models which corresponds to an approximation or reduction of the Hodgkin–Huxley equations, the Hindmarsh–Rose system is the result of biological experiments performed on the pond snails. Remarkably, this model has some important similarities with other neurons such as the S-shaped nullclines and the bistability property (see, e.g., Refs. 8–10 for detailed analyses). However, the second order model is not able to reproduce some interesting phenomena such as terminating by itself a triggered state of firing. Hence, to this aim the authors added a third dynamical component, whose role is to tune the above subsystem over the mono- and bistability regions in order to activate or termi-nate the neuronal response. Since such an additional term is designed to act on a longer time scale, the model results to be naturally divided into a fast and a slow subsystem. Notably, the recent paper Ref. 11 shows that the Hindmarsh–Rose neuron can capture both qualitative and quantitative aspects of experimental data.

Due to the particular structure of the model, several phenomena can be accurately described just by studying the fast subsystem, eventually taking care of some interactions with the slow component. However, this approach is not able to explain the complete variety of the behaviors exhibited by the model. To this aim the previous work Ref. 10 made an effort to provide a comprehensive description of the neuron phenomena in terms of its fully coupled equations, describing the generation of the different regimes in terms of bifurcation theory. In this paper we present the chaotic spiking-bursting transition as a specific expression of the whole third order dynamics. Section II introduces the mathematical model and its main properties. In Sec. III the most important features concerning the geometry of the bursting and spiking attractors are analyzed in details. Section IV introduces a dynami-
The third order Hindmarsh–Rose model equations are
\[
\begin{align*}
\dot{x} &= y - z + I + bx^2 - ax^3, \\
\dot{y} &= -y + c - dx^2, \\
\dot{z} &= rsx - rz - rsx_0,
\end{align*}
\]  
where \(x\) is the membrane potential, \(I\) is the external dc current, and \(y\) and \(z\) are the recovery and the adaptation current, respectively. The constant parameters \(a, b, c, \) and \(d\) define the underlying original second order system and the value \(x_0\) as well. In particular, they are set to the common values
\[
a = 1, \quad b = 3, \quad c = 1, \quad d = 5, \quad x_0 = -1.6.
\]
The variable \(z\) is the additional component and the coefficients \(r\) and \(s\) have the critical role of tuning the neuron response. For a proper choice of \(r\) and \(s\) the model is able to display both regular and chaotic bursting and firing as the current \(I\) varies. It is worth noticing that \(r\) is usually set to a very small value, so dividing the model into a fast subsystem (i.e., the second order model described by the first two equations) and a slow subsystem (the third equation).

The Hindmarsh–Rose neuron exhibits a chaotic spiking-bursting transition over a wide range of its parameters \(r\) and \(s\). In particular, this phenomenon has already been studied in the literature for \(r \in (0.001, 0.006)\) and \(s = 4\). Besides, the model has been extensively studied by varying \(I\) in Refs. 13 and 14, while two parameter analyses have been recently reported in Refs. 15 and 16, denoting the present interest for the subject.

Hereafter, we will assume \(r = 0.0021\) and \(s = 4\) in accordance to Ref. 10. For such values a chaos-chaos transition happens when \(I\) varies in the very narrow interval (3.29, 3.30), generally assumed as the passage between bursting and spiking (chaotic) regimes. This transition boils down into a sharp modification of the chaotic attractor size, as illustrated in Fig. 1. Since the main variation in the attractor size is observed along the \(z\) component, this can be properly illustrated by reporting the minimum of \(z\) versus the current \(I\) as in Fig. 2. To regard, Ref. 13 considers that such a geometric modification may be related to an interior crisis and due to the continuity with respect to the external current \(I\), this is called a “continuous internal crisis.” In Ref. 14 an extensive and quantitative analysis of the transition is provided according to the fast and slow subsystem division approaches. The phenomenon is strongly related to the pres-

![FIG. 1. (Color online) (a) A three-dimensional view of the chaotic spiking attractor for \(I = 3.30\) and its position with respect to the equilibrium point. (b) The chaotic bursting attractor and the equilibrium point for \(I = 3.29\), i.e., after the spiking-to-bursting chaotic transition described in Sec. II. The comparison between cases (a) and (b) shows the sharp modification of the attractor size along the \(z\) direction. Besides, the equilibrium point position has a negligible variation.]

![FIG. 2. (Color online) The variation in the attractor minimum value along the \(z\) direction as the current \(I\) varies in a narrow range around the transition interval. The related attractor modification appears to be continuous in nature.]

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ence in the fast subsystem of a homoclinic bifurcation and what happens in terms of the whole system dynamics is not investigated. A similar observation has already been expressed in Ref. 17. Moreover, this author introduces a different perspective as well. He suggests that the stable and unstable manifolds of the (unique) equilibrium point of the system may play an important role in determining the dynamics. Besides, the shape of the chaotic attractor may depend on the envelopment provided by such manifolds. According to this perspective, on one side of the transition interval the attractor is forced to stay in a certain region of the state space, while on the other side it is allowed to reach a further region, so modifying its shape.

III. THE TRANSITION BEHAVIOR

In this section we provide a detailed description of the neuron model (1) by illustrating the different shapes of the state space trajectories on the two sides of the chaotic spiking-bursting transition. Then, we will focus on the critical region where they differ to propose a dynamical explanation of the process.

Let us consider the system behavior for the decreasing values of the external current $I$ in the range (3.30, 3.29). Figure 1(a) illustrates the chaotic spiking attractor at $I = 3.30$. The trajectory turns along a spiral, where each coil nearly lies on a plane parallel to the $x$-$y$ subspace and moves having $z$ as dominant direction. When the trajectory reaches the top, a reinjection happens and the system state is rapidly brought back to the bottom. During this motion the $z$ dynamics is leading. Therefore, the chaotic spiking attractor results in a spiral component ruled by the fast subsystem dynamics and a reinjection path where the slow subsystem prevails.

In Fig. 1(b) the state space representation of the chaotic bursting attractor is depicted for $I = 3.29$. As in the previous case a spiral shape is recognized. The coils are almost parallel to the $x$-$y$ plane and their motion nearly follows the $z$ coordinate. The reinjection process can still be highlighted once the trajectory reaches the top of the spiral: while in the spiking case the paths are always similar, in the bursting case a variety of different routes are observed. In particular, they differ in the length, the dominant direction, and the $z$ heights to which the bring back the trajectory.

Summing up, when the state reaches the top of the spiral, it may undergo a reinjection process ranging between two main dynamics. In the first case a shorter path analogous to the spiking one is covered. In the second scenario the state takes a longer path. At the beginning, it assumes a different direction, except for a later turn, which gets the trajectory into the spiral at a lower $z$ height. In Fig. 3 a close look at the chaotic bursting attractor reveals that the critical region is at the spiral’s top, where the reinjection process is about to start. In particular, the way that the trajectories follow to approach such region affects how they are brought back to the bottom. A comparison between the paths highlighted in the figure reveals the different contributions to the attractor’s shape as well.

These observations and in particular the dominance of the $z$ dynamics along the reinjection process underline that the fast/slow subsystem technique is not suitable to explain the nature of the transition.

For the following developments we briefly recall the nature of the unique equilibrium point of Eqs. (1) and (2) during the transition (see Refs. 9 and 10 for further details). During the process such fixed point almost remains constant as well as its stability properties. Also the eigenvalues (two unstable and one stable) and the related eigenvectors undergo negligible variations and they can be considered as invariants along this passage. Although the equilibrium lies distant from the attractor, the direction of its eigenvectors indicates that it may play a fundamental role. Indeed, roughly speaking, on the side where the reinjection happens the spiral attractor is “squeezed” against the completely unstable eigenspace of the equilibrium (i.e., the plane tangent to its completely unstable manifold). Moreover, the trajectory approaches the starting region of the reinjection by a direction parallel to the stable eigenspace and the weakly unstable eigenspace turns out to pass close to that region too.

In Ref. 17 the author suggests that the shape of the attractor is due to the passage of the trajectory in two different regions of the state space. Furthermore, he chooses the plane formed by the stable and the weakly unstable eigenspaces of the equilibrium point to describe the related separatrix. This surface is just the linear approximation of the stable/weakly unstable manifold and it plays a passive role in the explanation of the phenomenon. Indeed, according to the author’s idea the dynamics may be completely attributed to only the fast subsystem. The Poincaré map built in Ref. 17 and derived for $r=0.0021$ in Fig. 4(a) provides just a simplified tool to represent the candidate separatrix, which turns out to detect the transition fairly well. However, this happens because such a map is able to catch the attractor’s size variation along $z$, as shown in Fig. 4(b). Indeed, moving from the spiking regime the transition is described in terms of the map’s intersection with the stable/weakly unstable line. However,
since the trajectories are not allowed to pass through a dynamical manifold, the situation depicted in Fig. 4(a) is not related to any real crossing phenomenon. Nevertheless, observing the corresponding positions in Fig. 4(b), it is straightforward to derive that such an event corresponds to a change in the attractor’s shape and that it occurs when the spiral’s bottom reaches a sufficiently low $z$. Since the transition is detected by the position of the attractor’s lowest part, such a Poincaré map appears not suitable to explain the behavior in the critical region at the top of the spiral, where really the spiking and bursting regimes are set up.

On the base of these observations, in Sec. IV we will introduce a novel conjecture for such a passage.

IV. THE CONJECTURE

From the analysis of Sec. III, the phenomenon can be treated according to the following conjecture. The reinjection process concerning the chaotic spiking-bursting transition occurs near the completely unstable manifold of the equilibrium point. In particular, the critical region at the top of the spiral is close to the weakly unstable component. Hence, in the first stage the reinjection is driven by such dynamics, while in the second stage the strongly unstable dynamics prevails. In the spiking case the trajectory approaches the completely unstable surface on the same side of the weakly unstable curve, while in the bursting case it may fall on both sides. Then, in the latter scenario the trajectory may diverge according to both directions of the strongly unstable component, resulting in two different reinjection paths.

In the following we address the considered transition more precisely by means of the scheme illustrated in Fig. 5. Let $E$ be the equilibrium point and $S$, $W$, and $U$ be its stable, weakly unstable, and strongly unstable manifolds, respectively. We denote by $U_1$ and $U_2$ the two branches of the curve $U$ from $E$ and by $\Phi$ the surface corresponding to the completely unstable manifold. Observe that $U$ and $W$ are curves on $\Phi$. Then, assume that the system trajectory approaches $\Phi$ approximately according to the direction of $S$.

The three different situations illustrated in Fig. 5 are considered.

(a) As the system trajectory approaches $\Phi$ the influence of the unstable dynamics of the equilibrium point becomes dominant. When the approaching region is sufficiently distant from $W$, the strongly unstable component prevails and the motion diverges along the branch $U_1$.

(b) The system trajectory tends to $\Phi$ very near to $W$. Hence, at the beginning the weakly unstable dynamics prevails and the motion diverges according to that curve. Then, the strongly unstable component necessarily dominates and the trajectory finally follows its dynamics, namely, along $U_1$.

(c) The system state approaches $\Phi$ close to the weakly unstable manifold, here on the opposite side of $W$ with respect to the case (b). The trajectory follows initially the curve $W$ but, as the strongly unstable dynamics becomes leading, it diverges along the opposite branch of $U$, namely, $U_2$.

Therefore, according to this scenario, the chaotic spiking is characterized at the spiral’s top by trajectories as that of case (b). Conversely, in the chaotic bursting the motion may exhibit in that critical region both the behaviors (b) and (c). In particular, case (b) is related to the shortest reinjection paths, while (c) is responsible for the longest ones.

Section V will be devoted to a rigorous analytical approach of these system dynamics in order to validate the above conjecture. Since the basic elements of such a phenomenon have been singled out in the scheme of Fig. 5, we will develop our approach for a more general class of systems, then deriving specific results for the Hindmarsh–Rose model.
V. THE MANIFOLD GEOMETRY AND THE TRANSITION

At first, we derive an approximation of the weakly unstable manifold $W$ to verify its presence in the critical region of the reinjection process. To this aim, we develop hereafter an analytical procedure designed for a more general class of dynamical systems, just presenting aside specific results for the Hindmarsh–Rose model.

Let us consider the class

$$\dot{\xi} = \Lambda \xi + \sum_{n=2}^{\infty} \gamma_n \delta^{[n]}(\xi),$$

where $\xi \in \mathbb{R}^N$, $\gamma_n \in \mathbb{R}^N$, $\delta^{[n]}(\xi) : \mathbb{R}^N \to \mathbb{R}$ is a homogeneous polynomial of the $n$th order in the components of $\xi$, and $\Lambda \in \mathbb{R}^{N \times N}$ is a diagonal matrix such that all its eigenvalues are negative but two. For the sake of simplicity we assume $N = 3$, so that

$$\xi = \begin{bmatrix} \xi_s \\ \xi_u \\ \xi_w \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_s & 0 & 0 \\ 0 & \lambda_u & 0 \\ 0 & 0 & \lambda_w \end{bmatrix}$$

with $\lambda_s < 0 < \lambda_u < \lambda_w$.

Let us prove that the Hindmarsh–Rose system admits model (3). To this aim first observe that it can be compactly written as

$$\dot{\xi} = \Lambda \xi + f(\xi)$$

just by defining

$$\xi = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ rs & 0 & -r \end{bmatrix}, \quad f(\xi) = \begin{bmatrix} I + bx^2 - ax^3 \\ -dx^2 \\ -rsx_0 \end{bmatrix}.$$

Moreover, let the equilibrium at a certain value of the external current $I$ be denoted as

$$E(I) = \begin{bmatrix} q_1(I) \\ q_2(I) \\ q_3(I) \end{bmatrix}.$$

For the sake of simplicity, we will not indicate further this dependence on $I$. Then, let $\lambda_s$, $\lambda_u$, and $\lambda_w$, respectively, be the stable, the strongly unstable, and the weakly unstable eigenvalues of $E$ at the considered value. In the transition interval they have the real values $\lambda_s = 0.1925$, $\lambda_u = 0.004366$, and $\lambda_w = -0.645$. Moreover, let $V_s, V_u, V_w \in \mathbb{R}^3$ be the three column eigenvectors related to $\lambda_s$, $\lambda_u$, and $\lambda_w$, respectively. Finally, define the modal matrix

$$T = [V_s, V_u, V_w] = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$$

and let us denote its inverse by

$$T^{-1} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}.$$

Then, we perform the linear affine change in coordinates

$$\xi = T^{-1}(\zeta - E),$$

obtaining
\[ \dot{\xi} = T^{-1}AT\xi + T^{-1}AE + T^{-1}f(T\xi + E). \]  
(7)

After tedious though straightforward computations Eq. (7) boils down to the system equation

\[ \dot{\xi}_s = \lambda_s \xi_s + (v_{11}(b - 3a\varrho_1) - d)v_{12}(u_{11}\xi_s + v_{12}\xi_u + v_{13}\xi_w)^2 \]
\[- av_{11}(u_{11}\xi_s + v_{12}\xi_u + v_{13}\xi_w)^3, \]
\[ \dot{\xi}_u = \lambda_u \xi_u + (v_{21}(b - 3a\varrho_1) - d)v_{22}(u_{11}\xi_s + v_{12}\xi_u + v_{13}\xi_w)^2 \]
\[- av_{21}(u_{11}\xi_s + v_{12}\xi_u + v_{13}\xi_w)^3, \]
\[ \dot{\xi}_w = \lambda_w \xi_w + (v_{31}(b - 3a\varrho_1) - d)v_{32}(u_{11}\xi_s + v_{12}\xi_u + v_{13}\xi_w)^2 \]
\[- av_{31}(u_{11}\xi_s + v_{12}\xi_u + v_{13}\xi_w)^3, \]  
(8)

in the new state variables \(\xi_s, \xi_u, \) and \(\xi_w\). Observe that Eq. (8) admits the representation (3) just by defining the functions

\[ g^{[1]}(\xi_s, \xi_u, \xi_w) = (v_{11}\xi_s + v_{12}\xi_u + v_{13}\xi_w)^2, \]
\[ g^{[2]}(\xi_s, \xi_u, \xi_w) = (v_{11}\xi_s + v_{12}\xi_u + v_{13}\xi_w)^3, \]
\[ g^{[3]}(\xi_s, \xi_u, \xi_w) = (v_{11}\xi_s + v_{12}\xi_u + v_{13}\xi_w)^4, \]
\[ g^{[n]}(\xi_s, \xi_u, \xi_w) = 0, \ \forall \ n > 3, \]  
(9)

and the vectors

\[ \gamma_2 = \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \end{bmatrix} = (b - 3a\varrho_1) \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} - \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}, \]
\[ \gamma_3 = \begin{bmatrix} \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix} = - \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}, \]  
(10)

To develop a suitable approximation of the weakly unstable manifold, let us recall that \(W\) is a curve passing through \(E\) with direction parallel to the eigenvectors of \(\lambda_w\). Now, system (3) has the origin as equilibrium point and the weakly unstable eigenvectors are directed along the axis \(\xi_w\). Therefore, \(W\) can locally be parametrized by the coordinate \(\xi_w\) as follows:

\[ W = \{ (\xi_s, \xi_u, \xi_w) \in \mathbb{R}^3; \xi_s = h_s(\xi_w), \xi_u = h_u(\xi_w) \}, \]  
(11)

where the functions \(h_s\) and \(h_u\) must satisfy the tangency conditions

\[ h_s(0) = h_u(0) = 0, \]
\[ \frac{\partial h_s}{\partial \xi_w}(0) = \frac{\partial h_u}{\partial \xi_w}(0) = 0, \]  
(12)

and they are the goal of our research. Let us represent \(h_s\) and \(h_u\) by power series developments

\[ h_s(\xi_w) = \sum_{n=2}^{\infty} a_n g_n^{[n]}(\xi_w), \]
\[ h_u(\xi_w) = \sum_{n=2}^{\infty} b_n g_n^{[n]}(\xi_w). \]  
(13)

Deriving a suitable approximation of \(W\) turns into the problem of computing the sets \(\{a_n\}_{n=2}^M\) and \(\{b_n\}_{n=2}^M\) for a sufficiently high number \(M - 1\) of elements, with a related error decreasing as \(M\) increases. To this aim, consider the nonlinear part of system (3) and evaluate it into the points of the manifold (11). Exploiting the representation (13) for \(h_s\) and \(h_u\), we obtain a \(\xi_w\) power series,

\[ \sum_{n=2}^{\infty} \gamma_n g_n^{[n]}(h_s(\xi_w), h_u(\xi_w), \xi_w) = \sum_{n=2}^{\infty} k_n g_n^{[n]}(\xi_w), \]

where the coefficients

\[ k_n = \begin{bmatrix} k_{1n} \\ k_{2n} \\ k_{3n} \end{bmatrix}, \]  
(14)

depend on the nature of the functions \(g_n^{[n]}\) and on the parameters \(\{a_n\}\) and \(\{b_n\}\) as well.

Remark: Each \(k_n\) only depends on \(a_{n-1}, b_{n-1}, \ldots, a_2, b_2\).

For the Hindmarsh–Rose system Eq. (14) reduces to

\[ \gamma_2 g^{[2]}(h_s(\xi_w), h_u(\xi_w), \xi_w) + \gamma_3 g^{[3]}(h_s(\xi_w), h_u(\xi_w), \xi_w) = \sum_{n=2}^{\infty} k_n g_n^{[n]}(\xi_w), \]  
(15)

where \(g^{[2]}\) and \(g^{[3]}\) and \(\gamma_2, \gamma_3\) are defined as in Eqs. (9) and (10), respectively. The expressions of the corresponding coefficients \(k_n\), \(i = 1, 2, 3\), are listed in Table I.

Substituting the power development (13) in the system equation (3) and exploiting expression (14) for the nonlinear term, we obtain

\[ \dot{\xi}_s(\xi_w) = \sum_{n=2}^{\infty} (a_n \lambda_s + k_{1n}) g_n^{[n]}(\xi_w), \]
\[ \dot{\xi}_u(\xi_w) = \sum_{n=2}^{\infty} (b_n \lambda_u + k_{2n}) g_n^{[n]}(\xi_w), \]
\[ \dot{\xi}_w(\xi_w) = \sum_{n=1}^{\infty} k_n g_n^{[n]}(\xi_w), \]  
(16)

where \(k_{3n} = \lambda_w\).

Since the manifold \(W\) is such that

\[ \dot{\xi}_s = \frac{\partial h_s}{\partial \xi_w}, \]
\[ \dot{\xi}_u = \frac{\partial h_u}{\partial \xi_w}, \]
\[ \dot{\xi}_w = \sum_{n=2}^{\infty} \left( g_n^{[n]} \sum_{m=2}^{n} (m\alpha_m k_{3(n-m+1)}) \right), \]  
(17)

by computing Eq. (17) with Eq. (13) and using the third equation in Eq. (16), we also obtain that

\[ \dot{\xi}_s(\xi_w) = \sum_{n=2}^{\infty} \left( g_n^{[n]} \sum_{m=2}^{n} (m\alpha_m k_{3(n-m+1)}) \right), \]
\[ \dot{\xi}_u(\xi_w) = \sum_{n=2}^{\infty} \left( g_n^{[n]} \sum_{m=2}^{n} (m\beta_m k_{3(n-m+1)}) \right), \]
\[ \dot{\xi}_w(\xi_w) = \sum_{n=2}^{\infty} \left( g_n^{[n]} \sum_{m=2}^{n} (m\beta_m k_{3(n-m+1)}) \right). \]  
(18)

Then, we equate Eq. (18) to the first two equations of Eq. (16) and we balance the \(\xi_w\) powers. Exploiting the above remark, the coefficients \(a_n, b_n\), which define \(W\) in Eqs. (11)–(13), are finally obtained as
Hindmarsh–Rose system obtained by using the coefficients in the critical region. Moreover, it highlights that the reinjection trajectories of the critical region, but now they may pass on both its sides. In particular, on one side the reinjection paths follow a trail analogous to the spiking regime ones, while on the other, they follow a different and longer route that reaches lower $z$ values.

In conclusion, let us define $d$ as the minimum distance between $\hat{W}$ and the system attractor $\Gamma$, i.e.,

$$d \equiv \min_{\eta \in \hat{W}} \eta - \xi_{\min}^2. \tag{21}$$

The analysis of $d$ provides useful insights about the influence of the weakly unstable manifold of the equilibrium point onto the neuron regime. To this aim, the diagram of $d$ as the external current $I$ varies is reported in Fig. 7. Notably, the distance almost vanishes in the chaotic bursting interval.

**Figure 6(b)** depicts the weakly unstable manifold approximation in comparison with the bursting chaotic attractor at $I=3.29$. The manifold $\hat{W}$ appears fully embedded in the trajectories of the critical region, but now they may pass on both its sides. In particular, on one side the reinjection paths follow a trail analogous to the spiking regime ones, while on the other, they follow a different and longer route that reaches lower $z$ values.

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Moreover, it has a minimum in the chaotic spiking-bursting transition window (3.29, 3.30), denoting the influence of the weakly unstable manifold on the system dynamics, in accordance with the conjecture of Sec. IV. By decreasing $I$ from higher values the model exhibits the chaotic spiking regime with trajectories passing on the same side of $\hat{W}$ and shows a continuous distance reduction. Then, in the neighborhood of the transition $d$ vanishes and the orbits tend to overcome $\hat{W}$, passing by both its sides during the chaotic bursting regime. When this behavior disappears, $d$ is again growing and several local minima are experienced for lower values of $I$ corresponding to other changes in the system dynamics, whose study is beyond the scope of the paper.

**VI. CONCLUSIONS**

The paper has introduced a novel explanation about the chaotic spiking-bursting transition in the Hindmarsh–Rose model. A qualitative analysis of the system behavior in the state space has highlighted that the fast/slow sub-system approach appears not suitable to comprehend the dynamical mechanism of the transition. Therefore, a detailed study of the trajectory evolution in the critical passage has been carried out, leading to formulate a conjecture about the influence of the equilibrium point onto the system regimes. Then, classical mathematical tools of nonlinear dynamics have been exploited to approximate the weakly unstable manifold of the fixed point as the significant element which governs the transition. Such an approach has been developed for a more general class of polynomial systems, deriving specific results for the Hindmarsh–Rose model. A numerical analysis has been performed to provide quantitative results as well. The accordance between such results and the conjectured behavior is remarkable and suggests the correctness of the dynamical explanation for the chaotic spiking-bursting transition. Moreover, the analysis points out that the considered phenomenon could be detected in other dynamical models, where sudden but continuous changes in the attractor shape occur.