Symmetric Overbounding of Correlated Errors

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ABSTRACT

In aviation navigation systems such as the Ground-Based and Space-Based Augmentation Systems (GBAS and SBAS) for GPS, it is critical for users to compute a conservative navigation-error bound during precision approach and landing. This paper describes a method for overbounding outputs of linear systems (i.e., conservatively describing their error distributions). Whereas earlier overbounding methods apply only to sets of independent samples, the methods developed in this paper handle samples drawn from correlated time series, so long as those samples have a probability distribution which a linear mapping can make spherically symmetric. Conservative error bounds constructed with these methods are useful in modeling the worst-case performance of filters under a wide range of environmental conditions. An example illustrates how symmetric bounding could enable aircraft to apply user-customized filtering to differential GPS corrections in GBAS or SBAS.

INTRODUCTION

Ground-Based and Space-Based Augmentation Systems (GBAS and SBAS) provide GPS users with high quality differential corrections that improve navigation accuracy. Depending on the configuration of the augmentation system, individual or networked reference stations make differential correction measurements and broadcast these to users through terrestrial VHF transmitters (in GBAS) or through a satellite link (in SBAS).

In addition to differential corrections, GBAS and SBAS messages also contain integrity information: both in the form of alert messages, which warn of possible satellite failures, and in the form of error parameters, which enable construction of conservative navigation-error bounds. Aviation users performing precision approach and landing operations rely on this integrity information to ensure they meet the safety requirements imposed by civil aviation authorities. In practice, users must define confidence bounds which bracket all but the rarest errors, those occurring with a risk of less than $10^{-7}$ (or $10^{-9}$ for fully automated landing). The user computes this confidence bound on-the-fly and compares the bound to a maximum allowable error for which the aircraft can still land within its touchdown box. As long as its confidence bound does not
exceed this limit, the aircraft can proceed safely with landing having validated that its navigation integrity risk meets the federally approved standard.

This operational test for integrity relies on conservatism in the user confidence bounds. To certify navigation hardware, manufacturers must demonstrate that their systems report a user error bound at least as large, or larger, than the actual value needed to protect the integrity risk requirement. Because the form of actual error distributions may vary in time or depend on environmental conditions, it is common to replace actual error distributions with conservative approximations, called overbounds. An overbound provides a conservative guarantee because its accumulated tail probability always exceeds the tail probability of the actual distribution.

This paper introduces methods to derive overbounds for the output of linear systems in airborne and other safety-critical applications. Linear systems, such as filters and estimators, commonly operate on signals corrupted by time-correlated measurement noise. These noise inputs may result from high-order and time-varying processes. Nonetheless, it may be desirable to describe the noise process with a lower-order, stationary model (such as a first-order Markov model) that is a conservative approximation of the actual noise process. The bounding strategy developed here, called symmetric overbounding, can replace time-varying and higher-order processes with stationary, lower-order approximations. Consequently, symmetric overbounding enables the description of noise inputs in a compact manner appropriate for communication over band-limited channels and for computation of real-time bounds on time-varying filters and estimators.

The first portion of this paper develops a quantitative basis for symmetric overbounding. Subsequently, the paper demonstrates the capabilities of symmetric overbounding by exploring the example of a variant GBAS architecture which would permit users to define their own linear filters for smoothing differential GBAS corrections.

**SYMMETRIC OVERBOUNDING**

This section develops a theory for overbounding the output of linear systems applied to signal inputs corrupted by correlated noise.

*Previous Research*

Previous research has established a set of guidelines for overbounding the error distribution that results from a sum (or a linear combination) of random variables. These operations are common in GPS, since user positioning errors equal the weighted sum of range measurement errors for each visible GPS satellite.
Currently, the certification of GPS augmentation systems relies heavily on Cumulative Distribution Function (CDF) overbounding. This technique constructs an overbound for a linear combination of random variables using individual overbounds for each input variable [1]. Although the original version of CDF overbounding applies only to independent, symmetric distributions with a single peak and zero-mean, other research has relaxed these restrictions. Paired-CDF overbounding and excess-mass overbounding, for instance, apply to arbitrary independent distributions, even those that are asymmetric, non-zero mean or have multiple peaks [2]-[3]. Because it is convenient to approximate actual distributions with Gaussians, other research has investigated the use of Gaussian bounds to represent independent distributions with heavier-than-Gaussian tails [4]-[6]. Additional work has addressed the topic of developing Gaussian overbounds from actual datasets [7],[8].

This paper extends earlier work by generalizing overbounding to apply to linear systems (of random signals) and not just to linear combinations (of random variables). Because random signals are frequently autocorrelated in time or cross-correlated with each other, these new bounding methods relax the independence constraint assumed in previous overbounding work.

**CDF Overbounding**

The basic notion of most overbounding techniques is to define an approximation that conservatively represents the Probability Distribution Function (PDF) for the output of an operation on one or more random variables. This approximation is conservative if its integrated tail probability everywhere exceeds the integrated tail probability of the actual output distribution. This definition can be quantitatively expressed as follows, where the overbar indicates an approximation and where the tilde indicates a dummy variable used for integration.

\[
\int_{\tilde{x}} \overline{p}(\tilde{x}) d\tilde{x} \geq \int_{x} p(\tilde{x}) d\tilde{x}, \quad x < x_{\text{med}} \\
\int_{\tilde{x}} \overline{p}(\tilde{x}) d\tilde{x} \geq \int_{x} p(\tilde{x}) d\tilde{x}, \quad x > x_{\text{med}} \tag{1}
\]

These relationships apply to the right and left distribution tails on either side of the distribution median, \(x_{\text{med}}\). Throughout this paper, the curved inequality symbol will imply the two overbound conditions given by (1). That is, the following notation indicates that the approximate distribution \(\overline{p}\) overbounds the actual distribution \(p\).

\[
\overline{p}(x) > p(x) \tag{2}
\]
Overbounding in this sense is sometimes called Cumulative Distribution Function (CDF) overbounding, because the conditions of (1) apply to the cumulative (integrated) probability in each tail of a PDF.

The earliest form of CDF overbounding was introduced in [1]. This form of CDF overbounding applies only to independent random variables. By contrast, the current paper develops a distinct form of CDF bounding that applies to correlated random variables with linearly mapped symmetric distributions. To distinguish between the two methodologies, this paper will refer to the earlier theory of [1] as independent-distribution overbounding and to the new form of CDF bounding introduced here as symmetric overbounding. Before introducing symmetric overbounding, it is useful to review the particulars of independent-distribution overbounding.

**Independent-Distribution Overbounding**

The independent-distribution overbounding theorem established in [1] states that linear combination operations preserve CDF overbounding. This theory applies to linear combinations of the form \( y \), which is the sum of two independent random variables \( x_a \) and \( x_b \) scaled by the deterministic coefficients \( \alpha_a \) and \( \alpha_b \).

\[
y = \alpha_a x_a + \alpha_b x_b
\]  

(3)

Reference [1] proves that the output distribution for \( y \) is overbounded by the convolution of the overbounds for \( x_a \) and \( x_b \), so long as both the overbounds and the actual error distributions are independent, symmetric, single-peaked and zero-mean. In symbolic form, this result may be written as

\[
\overline{p}_y(y) \succ p_y(y)
\]

(4)

when the approximation \( \overline{p}_y(y) \) is computed (after applying the multiplication rule described in the appendix) by the following equation in which the closed circle (\( \bullet \)) indicates the convolution operator,

\[
\overline{p}_y(y) = \frac{1}{|\alpha_a||\alpha_b|} \overline{p}_a\left(\frac{y}{|\alpha_a|}\right) \bullet \overline{p}_b\left(\frac{y}{|\alpha_b|}\right).
\]

(5)

given that

\[
\overline{p}_a(x_a) \succ p_a(x_a) \text{ and } \overline{p}_b(x_b) \succ p_b(x_b).
\]

(6)
Thus, under appropriate conditions, the PDF of the output of a linear combination may be computed conservatively by using conservative approximations of the input PDFs. The symmetry, single-peak, and zero-mean conditions are not generally significant when the noise is approximately zero-mean Gaussian; however, the requirement that the input random variables be independent places a significant constraint on many applications.

*Symmetric Overbounding*

This section develops a new CDF overbounding theorem that applies to linear combinations of random variables that are correlated. This method, called symmetric overbounding, requires that a linear transformation $G$ exist that maps the vector of correlated inputs $x$ to a space $z$ for which the joint PDF is spherically symmetric. Symbolically, symmetric overbounding places the following requirement on the joint PDF $p(x)$.

$$p(x \cdot x_0) = p(r), \quad r = \|z \cdot z_0\|_2 \quad z \cdot z_0 = G^{-1}(x \cdot x_0) \quad (7)$$

Here the mean of $x$ is $x_0$ and of $z$ is $z_0$. These mean values may be suppressed by substituting a new set of random variables for the originals ($x_{new} = x_{old} - x_0$). Thus, without loss of generality, distribution means will be dropped in the following analysis.

Given that the joint PDF for a set of random variables exhibits the symmetry implied by (7), it is possible to overbound the distribution of a linear combination of those variables $x$ by an appropriate scaling of the distribution for any individual element $x_i$.

$$\overline{p}_{x_i}(x_i) \triangleright p_{x_i}(x_i) \rightarrow \exists \xi \overline{p}_{\xi y}(\xi y) \triangleright p_y(y) \quad (8)$$

To prove this result and to obtain a value for the scaling factor $\xi$, consider a vector of random variables $x$ linearly combined to form an output $y$ using the deterministic weights contained in the vector $\alpha$.

$$y = \alpha^T x \quad (9)$$

The probability distribution for this combination can be expressed in terms of a linear mapping.

$$p_y(y) = p_y(\alpha^T Gz) = p_y(\alpha^T G \xi^T z). \quad (10)$$

Here a unit pointing vector has been introduced equal to
\[ e_j^\top = \frac{\alpha^\top G}{\|\alpha^\top G\|_2}. \]  

(11)

It is furthermore useful to define a scalar coordinate in the z-space along the line defined by the unit pointing vector \( e_j \).

\[ z_y = e_j^\top z. \]  

(12)

Applying the multiplication rule, described in the appendix, to (10) gives the following.

\[ p_y(y) = \frac{1}{\|\alpha^\top G\|_2} p_{x_j} \left( \frac{y}{\|\alpha^\top G\|_2} \right) = \frac{1}{\|\alpha^\top G\|_2} p_{z_i}(z_y) \]  

(13)

This distribution for the output \( y \) can be related to any one of the random inputs \( x_i \) using the z-space as an intermediary. The i-th random input \( x_i \) is related to the symmetrized vector \( z \) through the i-th row of the symmetrization matrix \( G \):

\[ x_i = G_i z. \]  

(14)

By analogy to (13), the distribution for \( x_i \) can be related to the z-space through the following equation.

\[ p_{x_i}(x_i) = \frac{1}{\|G_i\|_2} p_{z_i}(z_x). \]  

(15)

Here the variable \( z_x \) is defined to be the scalar coordinate projected along the axis of the vector \( G_i \). That is

\[ z_x = \frac{G_i}{\|G_i\|_2} z. \]  

(16)

Because the z-space probability is spherically symmetric, however, the marginal probability is identical when integrated onto any axis. Thus, spherical symmetry implies that

\[ p_{z_i}(z_x) = p_{z_j}(z_y) \]  

(17)

when \( z_x = z_y \). Using this symmetry relation, the probability distribution for \( y \) given by (13) can be related to the distribution for \( x \) given by (15).
\[ p_y(y) = \frac{\|G\|_{\text{2D}}}{\|\alpha^T G\|_{\text{2D}}} p_x \left( \frac{\|G\|_{\text{2D}}}{\|\alpha^T G\|_{\text{2D}}} y \right) \]  

Equation (18) indicates that the PDF for a linear combination is a scaled version of the PDF for any one of its inputs. This result is true so long as the joint distribution function for those random inputs is itself a linear mapping of a spherically symmetric distribution.

The scaling rule described by (18) can be used as the basis for overbounding any set of random variables \( x \) which can be linearly mapped into a spherically symmetric space. If any of the elements of \( x \) can be overbounded, such that

\[ \bar{p}_{x_i}(x_i) > p_{x_i}(x_i), \]  

then the linear combination \( y \) is, as a consequence of (18), overbounded by the following approximation,

\[ \bar{p}_y(y) = \frac{\|G\|_{\text{2D}}}{\|\alpha^T G\|_{\text{2D}}} \bar{p}_x \left( \frac{\|G\|_{\text{2D}}}{\|\alpha^T G\|_{\text{2D}}} y \right) > p_y(y). \]  

This completes the proof of the symmetric overbounding theorem, originally proposed as (8), and specifies the value of the scaling factor previously referred to as \( \xi \).

**Comparison of Bounding Methods**

It is instructive to compare the symmetric overbounding method with the original independent-distribution overbounding theorem. The linear map \( G \) that was introduced to decorrelate the random-variable inputs for symmetric overbounding can also be introduced to decorrelate the random-variable inputs for independent-distribution bounding. Specifically, this linear transformation \( G \) must map correlated random variables to a space in which their joint PDF is the product of several marginal PDFs.

\[ p_x(x) = \prod_j p_{z_j}(z_j), \quad z = G^{-1}x \]  

If such a linear map exists, then the independent-distribution overbound of (5) can be generalized to apply not only to independent random variables but also to correlated ones. In the case of the linear combination described by (9), the
generalized form of the independent distribution overbound is a sequence of convolutions between each of the N independent distributions in the mapped-variables space $z$.

$$\bar{p}_y(y) = \frac{1}{\alpha^T G_j} \bar{p}_{z_j} \left( \frac{y}{\alpha^T G_j} \right) \cdots \frac{1}{\alpha^T G_N} \bar{p}_{z_N} \left( \frac{y}{\alpha^T G_N} \right) \quad (22)$$

The overbound (22) is analogous to (20), where the linear map converts the input variables $x$ to a set of output variables $z$ that is independent, in the former case, and that is spherically symmetric, in the latter. Here it should be noted, in describing the generalized independent-distribution theorem, that the notation $G_j$ indicates a column vector whereas, in the description of symmetric overbounding, $G_i$ represents a row vector.

The overbounds produced by the symmetric and independent-distribution assumptions are distinct. The independent distribution theorem maps variables to a space in which they are uncorrelated and independent (in contrast with symmetric bounding for which the mapped variables are merely uncorrelated). Also, the independent-distribution method allows flexibility in that each random variable may possess a unique PDF (in contrast with symmetric bounding for which all random variables share PDFs of a common form). Most notably, however, the independent-distribution approach incurs significant computational expense. Compared to the symmetric bound of (20), for which the output is simply a stretched version of the input, the independent-distribution bound of (22) convolves together the PDFs for N inputs.

The computational load for repeated convolutions becomes especially significant when the number of input variables becomes very large. For moderate numbers of variables, efficient computation mitigates this computational load [9]. As N approaches infinity, however, repeated convolutions are infeasible unless the independent distributions are Gaussian (in which case the independent-distribution and symmetric overbounds are equivalent). Thus, the symmetric overbounding approach offers a practical solution to overbounding linear combinations with many correlated inputs, as long as those inputs are a linear mapping of a set of spherically symmetric random variables.

**Application to Linear Systems**

In this section, symmetric overbounding will be applied to bound the outputs of linear systems, such as filters and estimators. The inputs and outputs of linear systems are random signals rather than random variables. These signals represent sets of samples acquired discretely or continuously through time. Because an essentially infinite number of samples are processed, the low computational load of symmetric overbounding is critical.
Overbounding applies to linear systems because their outputs can be viewed as linear combinations of input samples weighted by system impulse functions. This connection is most clearly visible for Single-Input, Single Output (SISO) systems. The dynamics of any linear, time-varying SISO system are described by a function \( f(\Delta t, t) \) which describes the system’s response to an impulsive input at time \( t \). The noise output \( y(t) \) equals the inner product of the input noise \( v(t) \) with the reflected impulse response \( f \). For a system discretely sampled at an interval of \( T \) seconds, this inner product is the following.

\[
y(kT) = \sum_{m=0}^{\infty} f((k-m)T,kT) v(mT)
\]

\[
= f(0,kT)v(kT) + f(T,kT)v(kT-T) + ...
\]

Clearly, the output \( y \) is a linear combination of weighted input samples \( v \). Likewise in the continuous time case, when the sample interval \( T \) shrinks to zero, the output \( y \) is still a linear combination of input samples.

\[
y(t) = \int_{-\infty}^{\infty} f(\tau-t, t)v(\tau)\,d\tau
\]

No distinction need be drawn between the behavior of continuous and discretely sampled systems. Thus it is useful to introduce a generic convolution operator (\( \bullet \)) that takes the form of (23) for discretely sampled systems and of (24) for continuous time systems. For compactness, the arguments \( \Delta t \) and \( t \) will be suppressed unless they are specifically required for a derivation:

\[
y = f \bullet v .
\]

An equivalent Multiple Input, Multiple Output (MIMO) expression also exists. In the MIMO system, the inputs and outputs are vectors of signals. The matrix \( F \) contains the impulse functions relating each input in the vector \( v \) to an output in the vector \( y \).

\[
y = F \bullet v
\]

Here, the matrix convolution operator acts in an analogous manner to matrix multiplication, with convolutions substituted for element-wise products.
Because, by the analogy of (23), the above equation represents a form of linear combination, it is possible to apply symmetric overbounding. However, in order to apply symmetric overbounding, correlated elements of the input noise vector \( v \) must first be mapped to an uncorrelated, spherically symmetric form. It is possible to substitute a linear process \( G \) and a white noise input \( w \) for the correlated noise vector \( v \), as shown in Figure 1, if the linear process converts the white noise input into a correlated input with statistics matching those of \( v \).

\[
F \cdot v = \begin{bmatrix}
  f_{11}(t) & f_{12}(t) & \cdots & f_{1n}(t) \\
  f_{21}(t) & f_{22}(t) & \cdots & f_{2n}(t) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{m1}(t) & f_{m2}(t) & \cdots & f_{mn}(t)
\end{bmatrix} \cdot \begin{bmatrix}
  v_1(t) \\
  v_2(t) \\
  \vdots \\
  v_m(t)
\end{bmatrix}
\]

(27)

\[
= \begin{bmatrix}
  f_{11}(t) \cdot v_1(t) + f_{12}(t) \cdot v_2(t) + \cdots \\
  f_{21}(t) \cdot v_1(t) + f_{22}(t) \cdot v_2(t) + \cdots \\
  \vdots \\
  f_{m1}(t) \cdot v_1(t) + f_{m2}(t) \cdot v_2(t) + \cdots
\end{bmatrix}
\]

(27)

\[
v = G \cdot w
\]

(28)

Given precise knowledge of the noise process \( G \), the following symmetric overbound can be obtained.

\[
\bar{p}_{yi}(y_i(t)) = \frac{\|G_i(t)\|}{\|F \cdot G_i(t)\|} \bar{p}_{yi}\left(\frac{\|G_i(t)\|}{\|F \cdot G_i(t)\|} y_i(t)\right)
\]

(29)

\[
> p_{yi}(y_i(t))
\]

Figure 1. Overbounding Correlated Noise
This equation defines an overbound for the \( j \)-th element of the output noise vector in terms of the overbound for the \( i \)-th element of the input noise vector. Here \( G_i \) is the \( i \)-th row of the noise process matrix \( G \) and \( F_j \) the \( j \)-th row of the linear system impulse matrix \( F \). The linear system overbound (29) is analogous to the linear combination overbound (20) except that the linear system’s impulse matrix \( F \) replaces the linear combination’s weighting vector \( \alpha \). As in (20), the numerator represents the stretching associated with decorrelating the noise samples while the denominator represents the stretching associated with linearly combining those decorrelated noise samples.

To ensure that the 2-norm of (29) is well defined, this overbound is restricted to noise signals and impulse response functions in the \( L_2 \) signal space. The 2-norm of an \( N \)-element signal vector has the following form:

\[
\|G_i\|_2(t) = \sqrt{\sum_{k=1}^N \sum_{\Delta t} G^2_{ik}(\Delta t) d\Delta t}.
\]  

\[(30)\]

BOUNDING THE NOISE MAPPING FUNCTION

The previous section derived an equation to overbound the outputs of a linear system. This overbound, (29), relies strongly on precise knowledge of the mapping matrix \( G \), which shapes white noise to have statistical properties that match those of the correlated input noise. This section examines the noise mapping more closely, quantifying the relationship between \( G \) and the correlation of the input noise signals.

Subsequently, this section addresses the replacement of the actual map \( G \) with an approximation \( \tilde{G} \) (as shown in the bottom diagram of Figure 1). This replacement is necessary because \( G \) is a function of the input noise signals. In fact, it is not possible to determine \( G \) precisely because of statistical estimation error. Determination of \( G \) is especially challenging for nonstationary noise processes. By using a conservative, reduced-order approximation \( \tilde{G} \), however, it is possible to ensure bounding even when the estimate of the actual noise map \( G \) is uncertain.

Statistical Estimation of the Mapping Function

In order to apply the linear system overbound of (29), it is necessary to relate the noise map \( G \) to the correlation functions for the input noise signals. For this purpose, it is useful to define a correlation matrix \( Q \), in which each element is a correlation function belonging to the \( L_2 \) signal space. For a vector of random signals inputs \( v \), the corresponding correlation matrix is
denoted by a leading superscript: $^\prime Q$. The diagonal elements of $^\prime Q$ are the autocorrelation functions of the elements of $v$. Similarly, the off-diagonal elements of $^\prime Q$ are the cross-correlation functions between the elements of the signal vector $v$.

\[
^\prime Q = E\left( v \circ v^T \right) \tag{31}
\]

Here the open circle notation $(\circ)$ indicates a matrix correlation operation, analogous to the matrix convolution operation of (27).

The impulse-function matrix $G$ in the overbound (29) is directly related to this correlation matrix $^\prime Q$. In the statistical analysis of linear systems, it is well established that an autocorrelation function can be obtained by correlating the system’s impulse response with itself. A cross-correlation function can be obtained by correlating the impulse response functions of two different systems. Similarly, the elements of the correlation matrix $^\prime Q$ can be constructed from the impulse response elements of $G$ by substituting (28) into (31).

\[
^\prime Q = E\left( v \circ v^T \right) = E\left( (G \bullet w) \circ (w^T \bullet G^T) \right) \tag{32}
\]

This relationship is further simplified by noting that the white noise signals $w$ are not cross-correlated and that they are autocorrelated only at a zero time-shift ($\Delta t = 0$). The power in all the white-noise signals is assumed, without loss of generality, to have a constant value $c^2$.

\[
E\left( w_i \circ w_j \right) = \begin{cases} c^2 \delta(t) & i = j \\ 0 & i \neq j \end{cases} \tag{33}
\]

Evaluating (32) using the white-noise properties of (33) it is possible to express the correlation matrix $^\prime Q$ as the square of the impulse response matrix $G$.

\[
^\prime Q = c^2 \left( G \circ G^T \right) \tag{34}
\]

In practice, it is more convenient to estimate the correlation functions for a set of statistical data, rather than the impulse response function. For instance, if the noise process is slowly time varying, a statistical estimate of $^\prime Q$ is obtained by correlating the signals of the noise vector $v$ and averaging those correlation functions over $M + 1$ adjacent time samples.
\[
\hat{Q}_{i,j}(\Delta t, t) = \frac{1}{\Delta t} \sum_{m=-\Delta t}^{\Delta t} v_i(k-m) v_j(k-n-m) \\
\Delta t = nT, \quad t = kT
\]  

(35)

If the elements of the signal vector are the same \((i = j)\), this equation estimates their autocorrelation as a function of the time difference (\(nT\)) relative to the current time (\(kT\)). If the elements of the signal vector are not the same, the equation estimates their cross-correlation function.

Based on the elements of the correlation matrix, estimated through (35), and their relationship to the noise map, given by (34), it is possible to create an approximate overbound of the following form (where the hat superscripts indicate statistical approximations).

\[
\hat{p}_{ij}(y_j(t)) = \frac{\|\hat{G}_i(t)\|_2}{\|\hat{H}_j(t)\|_2} \frac{\|\hat{G}_j(t)\|_2}{\|\hat{H}_j(t)\|_2} y
\]  

(36)

Here \(H\) represents the total-system impulse response, which maps the white noise inputs through both the linear system \(F\) and the noise map \(G\).

\[
y = H \cdot w
\]  

(37)

\[
H = F \cdot G
\]  

(38)

The method for computing the approximate overbound, (36), will be discussed in more detail in the following section. The statistically estimated correlation matrix \(\hat{Q}\) does not necessarily guarantee conservative overbounding. Accordingly, a subsequent section discusses how to covert (36) from an approximate form to a guaranteed overbound.

*Approximate Overbounding Using Statistical Correlation*

After obtaining autocorrelation and cross-correlation estimates for each of the input signals in the vector \(v\) an approximate overbound can be constructed. In order to obtain the approximation (36), it is necessary to relate the 2-norms in the numerator and denominator of the equation to the correlation function estimates of (35). In establishing this relationship, it is useful to note that the value of the correlation matrix (31), evaluated at a zero time-shift (\(\Delta t = 0\), has the following form.
By comparing equation (39) to the 2-norm definition (30), it is evident that the diagonal elements of the correlation matrix are related to the 2-norms of each row of the impulse function matrix $G$ as follows.

\[
\|G_i\|_2 = c\sqrt{Q_{ii}(0,t_0)}
\]  

(40)

Using the estimated covariance of (35) in this equation results in a 2-norm estimate (denoted by a hat):

\[
\|\hat{G}_i\|_2 = c\sqrt{\hat{Q}_{ii}(0,t_0)}.
\]  

(41)

This statistical estimate approximates the numerator of the overbound (36). Computing the denominator requires further analysis. The 2-norm of the j-th row of the total system impulse response matrix $H$ can be related to the output variable correlation matrix by

\[
\|H_j\|_2 = c\sqrt{Q_{jj}(0,t_0)},
\]  

(42)

where the elements of the matrix $\gamma Q$ are the autocorrelation and cross-correlation functions of the output signal.

\[
\gamma Q = c^2 \left( H \cdot H^T \right).
\]  

(43)

The premise of linear system overbounding is that only the input noise signal for the linear system $F$ need be described. As a consequence, it should be possible to overbound the output noise of the system without actually measuring the statistics of the output signals. It is thus necessary to express the output correlation matrix $\gamma Q$ in terms of the input correlation matrix $\gamma Q$. Expanding the elements of $\gamma Q$ gives the following.

\[
\gamma Q_{ij} = H_i \cdot H_j^T
\]  

\[
= \sum_k \sum_l \sum_m \left( F_{i,l} \cdot G_{l,m} \right) \circ \left( G_{k,m} \cdot F_{j,k} \right)
\]  

(44)

Using the following identity

\[
(f \circ g)(f \circ g) = (f \circ f) \cdot (g \circ g),
\]  

(45)
the output variable correlation matrix $\gamma Q$, as described by (44), can be rewritten in terms of the input signal correlation matrix $\nu Q$.

\[
\gamma Q_{ij} = \sum_k \sum_l \left(F_{il} \circ F_{jk}\right) \bullet \sum_m \left(G_{lm} \circ G_{km}\right) \\
= \sum_k \sum_l \left(F_{il} \circ F_{jk}\right) \bullet \nu Q_{lk} \tag{46}
\]

Using this relationship, equation (42) can be rewritten as an estimate based on input vector statistics.

\[
\left\| \hat{H}_{ij} \right\|_2 = c\sqrt{\gamma \hat{Q}_{jj} (0, t_0)} \tag{47}
\]

\[
\gamma \hat{Q}_{ij} = \sum_k \sum_l \left(F_{il} \circ F_{jk}\right) \bullet \hat{Q}_{lk} \tag{48}
\]

Together equations (36), (41), and (47) provide a means to approximate the distribution for the outputs of a linear system. In contrast to the guaranteed overbound of (29), which assumed perfect knowledge of the noise mapping function $G$, the approximation distribution (36) is not guaranteed to be an overbound.

**Uncertainty in Characterizing the Noise Process**

This section describes why the approximate distribution of (36) is not guaranteed to overbound the output distribution for a linear system of interest. As an illustrative example, consider the greatly simplified case of a linear combination of two variables X and Y. For this case, it is assumed that the variables X and Y are correlated and that their joint PDF is spherically symmetric. A constant probability ellipse (solid curve) represents the “nominal” joint PDF in Figure 2.

Because statistical estimation is noisy, ellipses for the estimated correlation function may not match the nominal curve. As a result, linear combinations based on estimated joint PDFs may have more or less probability mass in their tails than the nominal PDF. Consider two estimates of the nominal constant-probability curve, one which is more positively correlated (dotted line) and one which is less positively correlated (dashed line). Linear combinations involving the nominal and approximate distributions are shown around the margin of Figure 2. For cases in which the linear-combination weights have the same sign, the more correlated approximation (dotted line) produces a conservative overbound but the less correlated approximation (dashed line) does not. The converse is true when the weights have opposite signs.
To guarantee overbounds for any linear combination (regardless of the signs of the weights $\alpha$) or for any linear system (regardless of the sign of the impulse functions $F$), it is critical to bracket uncertainty in the estimate of the noise map $G$. In graphical terms, the more and less positively correlated error ellipses of Figure 2 (dotted and dashed) bracket the actual error ellipse (solid curve). The following section quantifies this notion by developing a method to bracket the best available estimate $\hat{Q}$ of the correlation function matrix to account for estimation uncertainty.

**Conservatively Approximating the Noise Process Map**

In order to ensure a worst-case error bound that covers uncertainty (and nonstationary statistics), it is important to allow tolerance in the estimation of the correlation matrix $\hat{Q}$. This tolerance is labeled $Q'$ and represents the difference between the estimated and actual correlation matrices.

$$\hat{Q} = \hat{Q} + Q'$$

(49)

If the uncertain term $Q'$ can be bracketed by a reduced-order model, such as a first-order Markov process, then it is computationally efficient to incorporate the correlation uncertainty into an overbound, even if that overbound must be computed in real-time. This reduced-order model, which must upper and lower bound the uncertainty in $Q'$, is labeled $\Delta$.

$$\Delta_{i,j}(t) \geq c^{-2} |Q'_{i,j}(t)|$$

(50)
The elements of $\Delta$ are modeled correlation functions that bound uncertainty in the actual correlations. As long as the $\Delta$ matrix conservatively brackets $Q'$, an overbound can be guaranteed for the output of the linear system. This overbound takes the following form.

$$
\bar{p}_{ji}(y_j) = \left\| G \|_{L_2} \right\| \bar{p}_{ii}(y_j)\left( \left\| G \|_{L_2} \right\| y_j\right) > \left\| G \|_{L_2} \right\| \bar{p}_{ii}(y_j)\left( \left\| G \|_{L_2} \right\| y_j\right)
$$

(51)

Here the 2-norms that scale the overbounding distribution are replaced by a conservative overbound in the denominator (indicated by an overbar) and a conservative underbound in the numerator (indicated by an underbar).

It is straightforward to derive from (46), (49) and (50) that

$$
\bar{Q}_{sj} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left( F_{il} \circ F_{jk} \right) \bullet \hat{Q}_{lj} + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left( F_{il} \circ F_{jk} \right) \bullet \Delta_{lk} \geq \bar{Q}_{sj}
$$

(52)

and that an upper bound for the numerator of (51) is thus

$$
\left\| H \right\|_{L_2} = c\sqrt{\bar{Q}_{sj}(0, t_o)} \geq \left\| H \right\|_{L_2}.
$$

(53)

The absolute value of the second term of (52) ensures that the signs of the elements of the $\Delta$ matrix and the impulse functions of the $F$ matrix align to produce a worst-case bound.

A lower bound is required for the row-vector norm of $G$. A form of this lower bound can be obtained by substituting (49) into (40) and applying the bracketing relationship of (50). This time, rather than adding to the correlation matrix estimate to obtain an upper bound, the bracket matrix $\Delta$ is subtracted to obtain a lower bound.

$$
\bar{Q}_{oi}(0, t_o) = \max \left( \left| \hat{Q}_{oi}(0, t_o) \right| - \Delta_{oi}(0, t_o), 0 \right) \leq \bar{Q}_{oi}(0, t_o)
$$

(54)

$$
\left\| G \right\|_{L_2} = c\sqrt{\bar{Q}_{oi}(0, t_o)} \leq \left\| G \right\|_{L_2}
$$

(55)
A Procedure for Overbounding Linear Systems

The final form of the symmetric overbound for the output of a linear system is given by equation (51). Because this equation embeds several significant assumptions, it is useful to define a procedure that validates those assumptions prior to constructing the overbound. This procedure consists of five steps.

1. Inspect Spherical Symmetry Assumption

2. Define Overbound for any One Input Variable

3. Evaluate Statistical Correlation Functions

4. Estimate and Bracket all Correlation Functions

5. Compute Output Overbound Using (51)

Each of the steps of this procedure will be applied in the example of the following section.

BOUNDING A GPS DATASET

To demonstrate the principle of symmetric overbounding for a linear system, this section considers a GPS application. Specifically, symmetric overbounding is applied to demonstrate how aircraft could be permitted to define their own customized smoothing filters for the differential correction measurements broadcast by GBAS or SBAS.

In a conventional augmentation system, a set of ground stations generates differential correction measurements based on the known location for each reference antenna. The augmentation system filters the raw differential correction measurements and transmits the smoothed corrections to users. In standard GBAS systems, this filter is linear time-invariant with a 100-s time constant [10]. For various reasons, airborne users might benefit from applying longer or shorter smoothing times. Multiple smoothing rates, for example, could be applied to detect certain hazardous conditions such as the occurrence of an anomalous ionospheric storm [11].

Current GBAS and SBAS systems apply filtering to raw differential corrections at the ground station, rather than in the user avionics, for two primary reasons. First, differential corrections can be transmitted with higher precision if smoothing reduces the maximum and minimum values that must be transmitted. Second, the fact that the smoothing process is well defined permits the output noise to be characterized a priori based on accumulated statistics. If sufficient precision is
available to transmit raw corrections, then symmetric overbounding can alleviate this second consideration by providing a model of the raw differential corrections that enables output bounding, using (51), even if the form of the user’s measurement filter is not known to the ground station.

This section applies overbounds to a representative dataset to illustrate the potential for a modified GBAS with user-defined filtering of differential corrections. The example develops an input-noise bound for GPS measurements drawn from the National Geodetic Survey’s Continually Operation Reference Station (NGS-CORS) database [12]. The noise model derived from this dataset is applied to approximate the output of a set of low-pass smoothing filters with different time constants. The predicted output of the measurement filters, based on overbounding, is compared to the actual output when the filters are applied to the GPS data. The comparison validates the overbounds by demonstrating that the overbound tails always contain equal or greater probability than the tails of the actual output error distributions.

Data Set

The data analyzed in this section were acquired from a Wide Area Augmentation System (WAAS) reference station at the Oakland, California airport. This station is designated as ZOA1 in the NGS-CORS database. The measurement samples analyzed describe a 45 minute data record starting at a UTC hour of 0:00:00 on 28 August 2006. Samples were available at 1 s intervals.

Differential corrections can be generated for the ZOA1 data by subtracting the pseudorange measurements from the known range, based on survey, of the station relative to each GPS satellite. These differential corrections would be applied by nearby users to subtract out common-mode errors, such as ionosphere and troposphere delays that are highly correlated between users and the reference station.

The measurement error in the differential correction signal consists of noise that is not common-mode between the reference station and user. For the purposes of this paper, it is assumed that common mode errors change gradually over time. In fact, this assumption is quite reasonable under nominal conditions. As such, the uncorrelated measurement error is estimated as the residual after subtracting off a fourth-order polynomial fit from the ZOA1 range measurement data. The noise model developed in the remainder of this section describes these residual pseudoranges.
**User-Defined Pseudorange Smoothing**

In GBAS and SBAS, differential corrections are filtered with a digital low-pass filter that combines information about the underlying carrier signal with the code-phase ranging measurements [13],[14]. Often called a carrier-smoothing or Hatch filter, this class of filter uses low-noise carrier measurements of relative position to complement noisier code measurements of absolute position. The Hatch filter thus compensates for platform motion using relative ranging measurements provided by the carrier phase and enables long duration filtering of noisy, absolute range measurements provided by the code phase.

Under most conditions, the code-phase noise dominates that of the carrier-phase in the Hatch filter’s output error. Therefore it is representative to model only the noise associated with the code-phase pseudorange residuals. For this example, the code-phase noise will be modeled using the ranging residuals measured at ZOA1. Considering only code-phase errors, the following low-pass filter results.

\[
y(k) = \left[ \frac{T}{\tau} \left( 1 - \frac{T}{\tau} \right) \right] v(k) y(k-1) \tag{56}
\]

For standard GBAS and SBAS systems the sample interval \( T \) is 0.5 s and the time constant \( \tau \) is 100 s. As the data in this example were sampled at 1 Hz, however, \( T \) will be set to 1 s in this analysis. The impulse function for the low-pass filter of (56) is the following.

\[
f(k) = \frac{1}{\tau} (1 - \frac{T}{\tau})^k \tag{57}
\]

Because this filter is applied separately to each satellite ranging signal, this section will not consider cross-correlation among satellites and will therefore decouple the analysis of the smoothing filters for each ranging source. Consequently, the low-pass filter will be considered as a SISO system (25) rather than a MIMO system (26). Although this decoupling of ranging signals simplifies the example presented in this paper, cross-correlations between signals would need to be considered in a complete analysis of the navigation-error bound, as the GPS position solution combines the ranging measurements for all satellites.

**Step 1: Inspect Spherical Symmetry Assumption**

Because the number of input variables is essentially infinite, it is not possible to demonstrate formally that all inputs of a vector \( \mathbf{v} \) may be mapped to a set of spherically symmetric variables. Spherical symmetry must therefore be accepted as a
fundamental assumption. However, it is nonetheless prudent to inspect certain necessary conditions to verify qualitatively this assumption.

A reasonable check for spherical symmetry is to confirm visually that the autocorrelation and cross-correlation functions for each input signal are well behaved to first order. Specifically, a scatter plot can be generated for each input that compares signal values between subsequent epochs (autocorrelation check) or between pairs of input signals at the same epoch (cross-correlation check). Best fit ellipses of constant probability can then be added to the scatter plot. If the points in these plots appear uniformly distributed around the circumference of the best-fit elliptical bands, then the visual inspection verifies the symmetry assumption. If the points in the plot clearly skew or cluster and do not uniformly scatter around the circumference of the elliptical bands, then visual inspection contradicts the assumption of spherical symmetry. In this latter case, when the visual inspection yields a negative result, symmetric overbounding should not be applied to analyze the data set.

For the CORS range residuals from ZOA1, visual autocorrelation inspection does verify the symmetry assumption for all satellites. Figure 3 illustrates the symmetry inspection test for PRN 26. The axes of the scatter plot represent the values of the prompt residual \( v_k \) and the preceding residual \( v_{k-1} \) errors. The cross-covariance between these two sets of residuals was used to construct an inspection covariance matrix \( \mathbf{P} \). Ellipses of constant probability are plotted for four radii \( r = 1, 2, 3, 4 \) where the radius parameter is defined using the inspection covariance matrix as follows.

\[
    r = \sqrt{\begin{bmatrix} v_k & v_{k-1} \end{bmatrix} \mathbf{P}^{-1} \begin{bmatrix} v_k \\ v_{k-1} \end{bmatrix}} \quad (58)
\]

The density of the data points around each error ellipse contour appears to be approximately uniform. Consequently, the model of spherical symmetry (subject to a linear mapping) appears to be a reasonable representation of these data. Because the spherical symmetry assumption models the observed joint distribution for the measurement residuals well, the symmetric overbounding rule of (51) can be applied to this satellite.

The same qualitative test was repeated for each PRN in view. Because some satellites were subject to time-varying ranging residuals, however, the scatter plots for these satellites were generated over multiple shorter intervals of data (over which time-variation was minimal). With this caveat for time-variation, all satellites passed the visual inspection test for spherical symmetry.
Step 2: Define Overbound for any One Input Variable

The residual range errors for the dataset are well modeled by Gaussian distributions. Figure 3 shows an empirical CDF for each tail of the range residuals for PRN 26. The empirical CDF (solid line) is compared to a Gaussian CDF (dashed line) for which the standard deviation, sigma, has been inflated 3% relative to the sigma of the data. The inflated Gaussian CDF is clearly an overbound to the empirical data in the sense of the overbound definition (1). For the purposes of this example, the far tails of the actual error distribution are assumed also to be bounded by the inflated Gaussian CDF (dashed line). To validate this far tail bounding assumption would require the analysis of significantly more data collected over multiple days.

Similar Gaussian models bound all satellites in view during the 45 minute data window. However, several of the satellites exhibit significant nonstationarity. The Gaussian overbound was thus evaluated over only short duration windows to reduce the severity of time variation. Standard deviations of the data, $\hat{\sigma}_j$, (compiled for sequential 300 s windows) are illustrated in Figure 4 for each PRN j.
Step 3: Evaluate Statistical Correlation Functions

Constructing a symmetric overbound requires both that the distributions of the input variables are overbounded (Steps 1 and 2) and that the correlation functions for those variables are also appropriately bounded. For the present example which analyzes low-pass filters (acting independently on each ranging signal), only the autocorrelation functions for each satellite need be considered.

Because of nonstationarity in the input noise process, statistical autocorrelation functions were computed for each satellite over relatively-short, sequential windows of 300 s in duration. For this purpose, the correlation estimate equation (35) was used, with $M$ set to 300. Although the short length of the correlation window was necessary to capture nonstationary effects (such as phase shifts in oscillatory signals and time-variations in the noise power), a wider window would otherwise have been desirable to mitigate statistical sampling errors. As an alternative means of reducing sampling errors, these windowed autocorrelation functions were normalized to a peak value of one and averaged. Windows were not considered in the first and last 10 minutes of the data set, to provide margins for large-correlation time shifts, $\Delta t$. Excluding these points, five sequential data windows were defined for each satellite above the horizon at the start of the test (but only four windows were defined for PRN 4, which rose later). Statistically estimated correlation functions, averaged over all windows for each individual satellite, are plotted in Figure 5.

Figure 4. Standard Deviation of Range Residuals
After verifying spherical symmetry (Step 1), identifying an overbound for the input distribution (Step 2), and computing the autocorrelation functions for each satellite (Step 3), the critical step in developing the symmetric overbound (51) is the replacement of the autocorrelation function with a conservative, reduced-order model. This low-order model must also decompose the autocorrelation function into an estimate and a bracket on the deviation of the actual autocorrelation from that estimate, as described by (49).

The estimated autocorrelation need not be an unbiased estimate; in most cases, it is advantageous to define an upper and a lower bracket around the actual autocorrelation functions, as plotted in Figure 5 (dashed lines), and to set the estimate \( \hat{Q} \) equal to be the midpoint of the upper and lower bounds. For simplicity in this example, the estimated correlation function \( \hat{Q} \) is modeled as the correlation function for white noise: a delta function with a zero value for all time shifts away from \( \Delta t = 0 \).

The bracket \( \Delta \) around the estimate \( \hat{Q} \) is constructed from a composite of first-order Markov processes. The composite Markov model sums two first-order processes, as shown in Figure 6. In the figure, these processes are illustrated as a pair of mapping functions (\( \Lambda g \) and \( B g \)) each applied to a distinct Gaussian white noise input (\( Aw \) and \( Bw \)). The composite model, described by the following pair of impulse functions, is substituted for the input-noise signal \( v \) for each satellite \( j \).

\[
\Lambda g(k) = \beta_A (1 - \beta_A)^k \hat{\sigma}_j \sqrt{\rho} \\
B g(k) = \beta_B (1 - \beta_B)^k \hat{\sigma}_j \sqrt{1 - \rho}
\]

(59)

The parameters used to define this model were the inverse time-constants (\( \beta_A = 1/5 \) s\(^{-1} \) and \( \beta_B = 1/300 \) s\(^{-1} \)) and the power fraction (\( \rho = 1/36 \)) for the two Markov processes. These parameters were determined, by trial and error, to bound the autocorrelation functions empirically computed from the sampled data. The correlation functions for this composite Markov process are illustrated by the dashed curves in Figure 5, intended to represent upper and lower bounds on the empirical data.

The bracketing autocorrelation matrix \( \Delta \) thus maps two white noise signals \( w \) into a single correlated input \( v \). At a time shift of zero (\( \Delta t = 0 \)), the bracket is set to zero because the signal \( v \) is perfectly autocorrelated.

\[
\Delta(\Delta t, t) = \begin{cases} 
0 & \Delta t = 0 \\
(\Lambda g \circ \Lambda g + B g \circ B g) & \Delta t \neq 0
\end{cases}
\]

(60)
Figure 5 shows that the bracketing model bounds all of the averaged autocorrelation functions, except that of PRN 28. The autocorrelation function for PRN 28 is a special case, as sinusoidal oscillations are observed in the satellite-averaged correlations. These oscillations indicate strong multipath errors, which occur when GPS signals reflect from nearby surfaces and corrupt GPS receiver tracking of the direct signal.
Instead of extending the brackets to bound these multipath oscillations of PRN 28, an alternate solution is to define the estimate $\hat{Q}$ to incorporate the multipath process. Handling multipath in this manner presumes that the phenomenon may in fact be modeled as a random process. Such a model is appropriate because, with the exception of the strongest multipath waveforms that result from reflections off large planar surfaces (e.g. aircraft surfaces, buildings, or the ground), the occurrence of multipath is difficult to predict and model. Even when the occurrence of multipath can be predicted, the phase and amplitude of the periodic errors are generally not known, except in that they display moderate correlation from day to day over periods of one to two weeks. In this sense, multipath errors can be treated as the result of a noise process responding to a random input signal.

The behavior of the PRN 28 multipath noise is consistent with that of a second-order random process driven by white noise. The oscillation maintains a relatively consistent period, of 250 s in duration, but changes phase and amplitude infrequently. Breaking the autocorrelation windows into short records helps to reveal this periodic structure, as correlation effectively strips away phase information so that the windowed autocorrelations all align. Modeling the multipath noise process as a lightly-damped, second order impulse function results in an autocorrelation function that is cosinusoidal. This cosine is summed with a delta-function (white-noise) autocorrelation estimate for the remaining thermal noise and diffuse multipath. The combined estimated autocorrelation is the following.
\[
\hat{Q} = \begin{cases} 
\left( a \cos \left( \frac{\Delta t}{255} \right) + (1-a) \delta(\Delta t) \right) \hat{\sigma}_j^2 & \text{PRN 28} \\
\hat{\sigma}_j^2 \delta(\Delta t) & \text{PRN \neq 28}
\end{cases}
\] (61)

For PRN 28, the amplitude \(a\) of the cosine term can be determined by manual scaling.

Embedding the multipath ringing into the estimate of \(\hat{Q}\) reduces the residual error which the \(\Delta\) matrix must bound. The upper plot of Figure 7 shows the difference between the empirically computed autocorrelation function and the estimated autocorrelation for PRN 28 given by (61). This remainder is well bounded by the bracket function \(\Delta\). Thus, following multipath removal for PRN 28, the proposed \(\Delta\) model brackets the averaged correlation functions for all the satellites observed in the dataset (lower plot of Figure 7).

Figure 7. Impact of Multipath Decomposition

**Step 5: Compute Output Overbound**

The final remaining step is to compute the overbound of the output noise for one or more user filters. For this example, the user filter is assumed to take the form of the model first-order filter of (56) with a user-defined time constant \(\tau\). Because the form of the input overbound (from Step 2) was Gaussian, the output overbound is also Gaussian. However, the output distribution overbound is stretched according to (51). The degree of stretching depends on the 2-norm bounds \(\|g\|_2\) and \(\|h\|_2\).

According to (55), the value of this bound for \(\|g\|_2\) is determined by evaluating a combination of the estimated and bracketing autocorrelation functions at a time-shift of zero (\(\Delta t = 0\)). The bracket matrix \(\Delta\) has a zero value when \(\Delta t = 0\), as
specified by (60). The value of the estimated correlation function \( \hat{\mathbf{Q}} \) at a zero time shift is always equal to the variance \( \hat{\sigma}_j^2 \) of the overbounding Gaussian for any PRN \( j \). Thus, for the SISO filter considered, \( \| \mathbf{g} \|_2 = c \hat{\sigma}_j \).

The upper bound on the total-system impulse function \( \| \mathbf{h} \|_2 \) can be found using (52). Because the filter impulse function (57) is everywhere positive, the absolute value signs of (52) are not required. The resulting bound is

\[
\hat{\mathbf{Q}} = (f \circ f) \cdot (\hat{\mathbf{Q}} + \Delta).
\]  

Here \( f \) is given by (57), \( \hat{\mathbf{Q}} \) by (61) and \( \Delta \) by (60). Because the upper bound on the 2-norm of \( \mathbf{h} \) is

\[
\| \mathbf{h} \|_2 = c \sqrt{\hat{\mathbf{Q}}(0, t_0)},
\]  

the constant \( c \) cancels out of the stretching factor used to establish the symmetric overbound for the output \( \mathbf{y} \).

\[
\bar{p}_j (\mathbf{y}) = \frac{\| \mathbf{g} \|}{\| \mathbf{h} \|_2} \cdot \bar{p}_j \left( \frac{\| \mathbf{g} \|}{\| \mathbf{h} \|_2} \cdot \mathbf{y} \right) > p_j (\mathbf{y})
\]  

The output overbound thus has a Gaussian distribution that is inflated 3% (in Step 2) and by an additional scaling factor (in Step 5) equal to the ratio of the 2-norms bounds of \( \mathbf{g} \) and \( \mathbf{h} \).

Discussion

This section compares the output overbound, derived as (64), to the actual distributions that result from analyzing the statistics of the filter output signal. Because the data are Gaussian, these results are most easily compared through analysis of the standard deviation parameter, sigma. When two Gaussians are compared, the distribution with the larger sigma overbounds the distribution with the smaller sigma, according to the overbound definition (1).

By considering first-order filters of a range of different time constants, it is possible to validate the symmetric bound by showing that the computed overbound sigmas do in fact always exceed the sigmas that results when filters are applied to the actual data. Indeed, the overbounding method produces a conservative approximation of the output error for all nine satellites in the dataset. Figure 8 illustrates this result for three representative satellites: PRNs 4, 17, and 28.
When the input noise to the filter is essentially uncorrelated white noise (PRNs 4, 9, 11, 24, 27) the computed overbound is highly conservative. As an example, observe the case of PRN 4 shown in Figure 8. The sigma for the data is extremely close to the sigma value that would be expected for a white noise input. By comparison, the Markov model assumes significant correlation and so results in overconservatism.

The overbound sigma is a much tighter approximation of the data when the input noise is correlated (PRNs 8, 17, 26). As an example, consider the case of PRN 17 shown in Figure 8. The correlation function for this satellite drives the parameters that characterize the noise model (60). As a consequence, the overbounding sigma matches the data well. The bound is tightest

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**Figure 8. Applying Overbounds to Analysis of a Low Pass Filter**
for a time constant of approximately 20 s; as the time constant grows larger the bound drifts away from the data, indicating that the decay rate of the Markov process $B_g$ may be slightly slower and more conservative than necessary.

The overbound for the multipath affected satellite, PRN 28, is relatively tight. The data and overbound sigmas do drift apart rapidly at the left side of the plot (near $\tau = 0$) because of overconservatism in the $\Delta$ matrix. As the filter time constant increases, the more accurate multipath model dominates over the delta-function of (61). Consequently, the data and overbound sigmas roll off at the same rate across most of the range of filter time constants.

In order to achieve tighter bounds, it is clear that using distinct $\Delta$ bounds for each satellite would be beneficial. Also, it might be possible to tighten the bounds for a particular time-constant set point (i.e. at $\tau = 100$ s) by appropriately selecting the estimated and bracketing autocorrelation functions.

**CONCLUSIONS**

In the analysis of navigation systems for aviation, it is critical to represent the navigation error with a conservative bound. This paper developed a novel approach for overbounding output errors for linear systems that act on correlated input noise. The proposed bounding approach requires only that a linear transform can map the joint probability distribution function for the sampled data to a spherically symmetric form. This bound was applied to actual GPS data to illustrate the feasibility of a modified GBAS system that would broadcast raw differential corrections and allow users to define custom smoothing filters for those corrections.

**ACKNOWLEDGEMENTS**

This work was completed while J. Rife was a research associate with the Stanford GPS laboratory, Stanford, CA. The authors generously thank their colleagues in the Stanford GPS laboratory and their sponsors at the Federal Aviation Administration Satellite Navigation Program Office (AND-700) for supporting this research. The opinions discussed here are those of the authors and do not necessarily represent those of the FAA or other affiliated agencies.

REFERENCES


This appendix briefly reviews the multiplication of a random variable by a scalar. Suppose that the probability distribution of a variable $x$ is $p_x(x)$. If a second random variable $y$ is formed through scalar multiplication (i.e. $y = ax$), then the resulting probability distribution, $p_y(y)$, may be determined from the original distribution for $x$ using the following equation.

$$
p_y(y) = \frac{1}{a} p_x\left(\frac{y}{a}\right) \tag{65}
$$

This result is easily proved by noting that the probability of $\{y < Y\}$ equals the probability that $\{x < Y/a\}$. Equivalently, it can be stated that the cumulative distribution function of $y$, $P_y(y)$, equals the cumulative distribution function of $x$, $P_x(x)$, at the point where the value $y$ is the scaled value of $x$.

$$
P_y(y) = P_x\left(\frac{y}{a}\right) \tag{66}
$$

Since the probability distribution function is the derivative of the cumulative distribution, (65) is obtained simply by taking the derivative of (66). If the distribution of $p_x(x)$ is symmetric, then it is easy to demonstrate that the distribution $p_y(y)$ depends only on the magnitude of $a$. That is:

$$
p_y(y) = \frac{1}{|a|} p_x\left(\frac{|y|}{|a|}\right). \tag{67}
$$