The automorphism group of the free group of rank two is a CAT(0) group

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Abstract

We prove that the automorphism group of the braid group on four strands acts faithfully and geometrically on a CAT(0) 2-complex. This implies that the automorphism group of the free group of rank two acts faithfully and geometrically on a CAT(0) 2-complex, in contrast to the situation for rank three and above.

1 Introduction

A CAT(0) metric space is a proper complete geodesic metric space in which each geodesic triangle with side lengths $a$, $b$ and $c$ is “at least as thin” as the Euclidean triangle with side lengths $a$, $b$ and $c$ (see [5] for details). We say that a finitely generated group $G$ is a CAT(0) group if $G$ may be realized as a cocompact and properly discontinuous subgroup of the isometry group of a CAT(0) metric space $X$. Equivalently, $G$ is a CAT(0) group if there exists a CAT(0) metric space $X$ and a faithful geometric action of $G$ on $X$. It is perhaps not standard to require that the group action be faithful, a point which we address in Remark 1 below.

For each integer $n \geq 2$, we write $F_n$ for the free group of rank $n$ and $B_n$ for the braid group on $n$ strands.

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In [3], T. Brady exhibited a subgroup \( H \leq \text{Aut}(F_2) \) of index 24 which acts faithfully and geometrically on \( \text{CAT}(0) \) 2-complex. In subsequent work [4], the same author showed that \( B_4 \) acts faithfully and geometrically on a \( \text{CAT}(0) \) 3-complex. It follows that \( \text{Inn}(B_4) \) acts faithfully and geometrically on a \( \text{CAT}(0) \) 2-complex \( X_0 \) (this fact is explained explicitly by Crisp and Paoluzzi in [8]). Now, \( \text{Inn}(B_n) \) has index two in \( \text{Aut}(B_n) \) [10], and \( \text{Aut}(F_2) \) is isomorphic to \( \text{Aut}(B_4) \) [16, 10], thus the result in the title of this paper is proved if we exhibit an extra isometry of \( X_0 \) which extends the faithful geometric action of \( \text{Inn}(B_4) \) to a faithful geometric action of \( \text{Aut}(B_4) \). We do this in §2 below.

Our result reinforces the striking contrast between those properties enjoyed by \( \text{Aut}(F_2) \) and those enjoyed by the automorphism groups of finitely generated free groups of higher ranks. We can now say that \( \text{Aut}(F_2) \) is a \( \text{CAT}(0) \) group, a biautomatic group [11, 17] and it has a faithful linear representation [9, 16]; while \( \text{Aut}(F_n) \) is not a \( \text{CAT}(0) \) group [13], nor a biautomatic group [6] and it does not have a faithful linear representation [12] whenever \( n \geq 3 \).

We regard the \( \text{CAT}(0) \) 2-complex \( X_0 \) as a geometric companion to Auter Space (of rank two) [14], a topological construction equipped with a group action by \( \text{Aut}(F_2) \).

Let \( W_3 \) denote the universal Coxeter group of rank 3— that is, \( W_3 \) is the free product of 3 copies of the group of order two. Since \( \text{Aut}(F_2) \) is isomorphic to \( \text{Aut}(W_3) \) (see Remark 2 below), we also learn that \( \text{Aut}(W_3) \) is a \( \text{CAT}(0) \) group.

**Remark 1.** As pointed out in the opening paragraph, our definition of a \( \text{CAT}(0) \) group is perhaps not standard because of the requirement that the group action be faithful. We note that such a requirement is redundant when giving an analogous definition of a word hyperbolic group. This follows from the fact that word hyperbolicity is an invariant of the quasi-isometry class of a group. In contrast, the \( \text{CAT}(0) \) property is not an invariant of the quasi-isometry class of a group. Examples are known of two quasi-isometric groups, one of which is \( \text{CAT}(0) \), and the other of which is not. Examples of this type may be constructed using the fundamental groups of graph manifolds [15] and the fundamental groups of Seifert fibre spaces [5, p.258][1]. So the adjective ‘faithful’ is not so easily discarded in our definition of a \( \text{CAT}(0) \) group. We do not know of two abstractly commensurable groups, one of which is \( \text{CAT}(0) \), and the other of which is not. We promote the following question.
**Question 1.** Is the property of being a CAT(0) group an invariant of the abstract commensurability class of a group?

Some relevant results in the literature show that two natural approaches to this question do not work in general. If $G$ acts geometrically on a CAT(0) space $X$ and $G'$ is a finite extension with $[G' : G] = n$, then $G'$ acts properly and isometrically on the CAT(0) space $X^n$ with the product metric [19] [7, p.190]. However, proving this action is cocompact is either difficult or impossible in general. In [2], the authors give examples of the following type: $G$ is a group acting faithfully and geometrically on a CAT(0) space $X$, $G'$ is a finite extension of $G$, yet $G'$ does not act faithfully and geometrically on $X$. However, $G'$ may act faithfully and geometrically on some other CAT(0) space.

**Remark 2.** The fact that Aut($F_2$) is isomorphic to Aut($W_3$) appears to be well-known in certain mathematical circles, but is rarely recorded explicitly. We now outline a proof: the subgroup $E \leq W_3$ of even length elements is isomorphic to $F_2$, characteristic in $W_3$ and $C_{W_3}(E) = \{1\}$; it follows from [18, Lemma 1.1] that the induced homomorphism $\pi : \text{Aut}(W_3) \rightarrow \text{Aut}(E)$ is injective; one easily confirms that the image of $\pi$ contains a set of generators for $\text{Aut}(E)$, and hence $\pi$ is an isomorphism. A topological proof may also be constructed using the fact that the subgroup $E$ of even length words in $W_3$ corresponds to the 2-fold orbifold cover of the the orbifold $S^2(2, 2, 2, \infty)$ by the once-punctured torus.

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## 2 Aut($B_4$) is a CAT(0) group

We shall describe an apt presentation of $B_4$, give a concise combinatorial description of Brady’s space $X_0$, describe the faithful geometric action of Inn($B_4$) on $X_0$ and, finally, introduce an isometry of $X_0$ to extend the action of Inn($B_4$) to a faithful geometric action of Aut($B_4$).

The interested reader will find an informative, and rather more geometric, account of $X_0$ and the associated action of Inn($B_4$) in [8].

**An apt presentation of $B_4$:** A standard presentation of the group $B_4$ is

$$\langle a, b, c \mid aba = bab, bcb = cbc, ac = ca \rangle. \quad \text{(1)}$$
Introducing generators $d = (ac)^{-1}b(ac)$, $e = a^{-1}ba$ and $f = c^{-1}bc$, one may verify that $B_4$ is also presented by
\[
\langle a, b, c, d, e, f \mid ba = ac = eb, de = ec = cd, bc = cf = fb, \]
\[df = fa = ad, ca = ac, ef = fe \rangle.
\]
We set $x = bac$ and write $\langle x \rangle \subset B_4$ for the infinite cyclic subgroup generated by $x$. The center of $B_4$ is the infinite cyclic subgroup generated by $x^4$.

**The space $X_0$:** Consider the 2-dimensional piecewise Euclidean CW-complex $X_0$ constructed as follows:

(0-S) the vertices of $X_0$ are in one-to-one correspondence with the left cosets of $\langle x \rangle$ in $B_4$—we write $v_{g(x)}$ for the vertex corresponding to the coset $g\langle x \rangle$;

(1-S) distinct vertices $v_{g_1(x)}$ and $v_{g_2(x)}$ are connected by an edge of unit length if and only if there exists an element $\ell \in \{a, b, c, d, e, f\}^\pm$ such that $g_2^{-1}g_1\ell \in \langle x \rangle$;

(2-S) three vertices $v_{g_1(x)}$, $v_{g_2(x)}$ and $v_{g_3(x)}$ are the vertices of a Euclidean (equilateral) triangle if and only if the vertices are pairwise adjacent.

The link of the vertex $v_{(x)}$ in $X_0$, just like the link of each vertex in $X_0$, consists of twelve vertices (one for each of the cosets represented by elements in $\{a, b, c, d, e, f\}^\pm$) and sixteen edges (one for each of the distinct ways to spell $x$ as a word of length three in the alphabet $\{a, b, c, d, e, f\}$—see [8] for more details). It can be viewed as the 1-skeleton of a Möbius strip. In Figure 1 we depict the infinite cyclic cover of the link of $v_{(x)}$. Each vertex with label $g$ in the figure lies above the vertex $v_{g(x)}$ in the link of $v_{(x)}$. The link is formed by identifying identically labeled vertices and identifying edges with the same start and end points.

That $X_0$ is CAT(0) follows most naturally from the alternative construction of $X_0$ described in detail in [8]. Alternatively, a complex constructed from isometric Euclidean triangles is CAT(0) if and only if it is simply-connected and satisfies the ‘link condition’ [5, Theorem II.5.4, pp.206]. For a 2-dimensional complex, the link condition requires that each injective loop in the link of a vertex has length at least $2\pi$, where edges in a link are assigned the length of the angle they subtend [5, Lemma II.5.6, pp.207]. It is easily seen that $X_0$ satisfies the link condition because each injective loop in Figure
1 crosses at least 6 edges and each edge has length $\pi/3$. Thus one might show that $X_0$ is CAT(0) by showing that it is simply-connected. We shall not digress from the task at hand to provide such an argument.

**Brady's faithful geometric action of $\text{Inn}(B_4)$ on $X_0$:** We shall describe Brady's faithful geometric action of $\text{Inn}(B_4)$ on $X_0$. We shall do so by describing an isometric action $\rho : B_4 \to \text{Isom}(X_0)$ such that the image of $\rho$ is a properly discontinuous and cocompact subgroup of $\text{Isom}(X_0)$ which is isomorphic to $\text{Inn}(B_4)$.

It follows immediately from (1-S) that, for each $g \in B_4$, the “left-multiplication by $g$” map on the 0-skeleton of $X_0$, $g_1 \langle x \rangle \mapsto g g_1 \langle x \rangle$, extends to a simplicial isometry of the 1-skeleton of $X_0$. It follows immediately from (2-S) that any simplicial isometry of the 1-skeleton of $X_0$ extends to a simplicial isometry of $X_0$. We write $\phi_g$ for the isometry of $X_0$ determined by $g$ in this way, and we write $\rho : B_4 \to \text{Isom}(X_0)$ for the map $g \mapsto \phi_g$. We compute that $\rho(g_1 g_2)(v_{g_1}(x)) = v_{g_1 g_2 g_2}(x) = \rho(g_1)\rho(g_2)(v_{g_2}(x))$ for each $g_1, g_2, g \in B_4$, so $\rho$ is a homomorphism. Further, $\phi_g(v_{g}(x)) = v_{g}(x)$ for each $g \in B_4$, so the vertices of $X_0$ are contained in a single $\rho$-orbit. It follows that $\rho$ is a cocompact isometric action of $B_4$ on $X_0$.

To show that the image of $\rho$ is isomorphic to $\text{Inn}(B_4)$, it suffices to show that the kernel of $\rho$ is exactly the center of $B_4$. One easily computes that $\rho(x^4)$ is the identity isometry of $X_0$. Thus the kernel of $\rho$ contains the center of $B_4$. It is also clear that the stabilizer of $v_{(x)}$, which contains the kernel of $\rho$, is the infinite subgroup $\langle x \rangle$. So to establish that the kernel of $\rho$ is exactly the center of $B_4$, it suffices to show that $\phi_x, \phi_{x^2}$ and $\phi_{x^3}$ are non-trivial and distinct isometries of $X_0$. We achieve this by showing that these elements act non-trivially and distinctly on the link of $v_{(x)}$ in $X_0$. We compute that $x$ acts as follows on the cosets corresponding to vertices in the link of $v_{(x)}$. 

![Figure 1: A covering of the link of $v_{(x)}$ in $X_0$.](image)
where $\delta = \pm 1$:

\[
\begin{align*}
a^\delta \langle x \rangle &\mapsto e^\delta \langle x \rangle &\mapsto e^\delta \langle x \rangle &\mapsto f^\delta \langle x \rangle &\mapsto a^\delta \langle x \rangle &\text{and } b^\delta \langle x \rangle &\mapsto d^\delta \langle x \rangle.
\end{align*}
\]

Thus the restriction of $\phi_x$ to the link of $v(x)$ may be understood, with reference to Figure 2, as translation two units to the right followed by reflection across the horizontal dotted line. It follows that $\phi_x, \phi_{x^2}, \phi_{x^3}$ are non-trivial and distinct isometries of $X_0$, as required.

We next show that the image of $\rho$ is a properly discontinuous subgroup of $\text{Isom}(X_0)$. Now, the action $\rho$ is not properly discontinuous because, as noted above, the $\rho$-stabilizer of $v(x)$ is the infinite subgroup $\langle x \rangle$ (so infinitely many elements of $B_4$ fail to move the unit ball about $v(x)$ off itself). But the image of $\langle x \rangle$ under the map $B_4 \to \text{Inn}(B_4)$ has order four. It follows that the image of $\rho$ is a properly discontinuous subgroup of $\text{Isom}(X)$.

Thus we have that the image of $\rho$ is a properly discontinuous and cocompact subgroup of $\text{Isom}(X_0)$ which is isomorphic to $\text{Inn}(B_4)$.

**Extending $\rho$ by finding one more isometry:** It was shown in [10] that the unique non-trivial outer automorphism of $B_4$ is represented by the automorphism which inverts each of the generators in Presentation (1). Consider the automorphism $\tau \in \text{Aut}(B_4)$ determined by

\[
\begin{align*}
a &\mapsto a^{-1}, & b &\mapsto d^{-1}, & c &\mapsto c^{-1}, & d &\mapsto b^{-1}, & e &\mapsto f^{-1}, & f &\mapsto e^{-1}.
\end{align*}
\]

Note that $\tau$ is achieved by first applying the automorphism which inverts each of the generators $a, b$ and $c$ and then applying the inner automorphism $w \mapsto (ac)^{-1}w(ac)$ for each $w \in B_4$. It follows that $\tau$ is an involution which represents the unique non-trivial outer automorphism of $B_4$. Writing $J := B_4 \rtimes \mathbb{Z}_2$, we have $\text{Aut}(B_4) \cong J/\langle x^4 \rangle$. We identify $B_4$ with its image in $J$.

The automorphism $\tau \in \text{Aut}(B_4)$ permutes the elements of $\{a, b, c, d, e, f\}^{\pm 1}$ and maps the subgroup $\langle x \rangle$ to itself (in fact, $\tau(x) = x^{-1}$). It follows from (1-S) that the map $v_{g(x)} \mapsto v_{\tau(g)(x)}$ on the 0-skeleton of $X_0$ extends to a simplicial isometry of the 1-skeleton of $X_0$, and hence also to a simplicial isometry $\theta$ of $X_0$. We compute that $\theta \phi_g \theta = \phi_{\tau(g)}$ for each $g \in B_4$. Thus we have an isometric action $\rho : J \to \text{Isom}(X_0)$ given by

\[
g \mapsto \phi_g \text{ for each } g \in B_4, \text{ and } \tau \mapsto \theta.
\]

We also compute that the restriction of $\theta$ to the link of $v(x)$ may be understood as reflection across the vertical dotted line shown in Figure 2. It follows that
Figure 2: A covering of the link of the vertex $v_{(x)}$ and the fixed point sets of some reflections.

$\theta$ is a non-trivial isometry of $X_0$ which is distinct from $\phi_x, \phi_{x^2}$ and $\phi_{x^3}$. Thus the kernel of $\rho'$ is still the center of $B_4$, and the image of $\rho'$ is a properly discontinuous and cocompact subgroup of $\operatorname{Isom}(X_0)$ which is isomorphic to $\operatorname{Aut}(B_4)$. Hence we have a faithful geometric action of $\operatorname{Aut}(B_4)$ on $X_0$, as required.

References


