THE ANGLE QUESTION

KIM E. RUANE

ABSTRACT. The main result of this paper concerns CAT(0) groups $\Gamma$ which contain an infinite order element in the center. It suffices to consider the case $\Gamma = G \times \mathbb{Z}$ here. We show that although the $G$-factor does not determine a quasi-convex subset inside the space $\Gamma$ is acting on, we do show that there is a well-defined angle which, for a given action, describes how the $G$-factor sits inside the space.

INTRODUCTION

The angle question arose while studying CAT(0) groups of the form $G \times \mathbb{Z}$ with $G$ word hyperbolic. A CAT(0) group is a group which acts geometrically (isometrically, properly discontinuously, and cocompactly) on a CAT(0) space. In section two we mention an example of this form where the $G$ factor does not determine a quasi-convex subset of the space the group is acting on, however there is a well-defined angle which describes how $G$ sits inside the space.

The techniques apply to any CAT(0) group which contains an infinite order element in the center. This is because any such element virtually breaks off as a direct factor from the group. Since we are considering questions concerning the boundary, we can assume the group actually splits off a $\mathbb{Z}$ factor. The necessary facts to make this precise are stated in the first section as well as some other necessary definitions and known results.

Section two contains the example mentioned above and the angle calculation is done there. In fact, we point out here that the only action of this group on a CAT(0) space which does determine a quasi-convex subset is the product action and the angle is $\frac{\pi}{2}$ in that case. Before proceeding to the proof of the main theorem, we must discuss how to measure angles between points of the boundary. This is the content of section three, where we again use [BH] as our reference.

The main result of this paper is obtained by analyzing how the limit points of $G$ sit inside $\partial X$ where $X$ is the space $G \times \mathbb{Z}$ is acting on geometrically. In this case, it is known that $\partial X$ is of the form $\Sigma(\partial Y)$ where $Y$ is a CAT(0) subspace of $X$ and $\Sigma(\partial Y)$ denotes the suspension of the boundary of $Y$ (see section 1 for definition). In the case that $G$ is word hyperbolic, we can say a bit more - precisely, $Y$ is quasi-isometric to $G$ which implies $\partial X = \Sigma(\partial G)$. The two main results are proved in section 4.

1991 Mathematics Subject Classification. Primary: 20F32; Secondary: 20H15.

Key words and phrases. word hyperbolic groups, CAT(0) spaces, boundaries of groups, angle metric.

*Supported by NSF grant DMS-97-04939
Theorem 4.2. Let $\Theta(G) = \inf \{ \angle(g, \gamma) : g \in G, \ o(g) = \infty \}$ be the angle of $G$ inside $X$. Then $\Theta(G) > 0$ - in particular, no rational ray can limit to a suspension point.

Theorem 4.5. There exists a constant $0 \leq B < \frac{\pi}{2}$ such that $\mathcal{L}(G)$ is contained in the $[-B, B]$ interval of the suspension $\partial X = \Sigma(\partial Y)$. Explicitly, for $z \in \mathcal{L}(G)$, $z = [z', \theta] \in \Sigma(\partial Y)$ where $z' \in \partial Y$ and $|\theta| < B$.

I would especially like to thank the referee for pointing out a mistake in the original version of the proof of Theorem 4.2.

**Basics on Isometries of CAT(0) Spaces**

In this section we give definitions and basic properties of CAT(0) spaces, boundaries and isometries as well as some known facts we will need in the proof of the main result.

Let $(X, d)$ be a metric space. Then $X$ is proper if metric balls are compact. A geodesic from $x$ to $y$ for $x, y \in X$ is a map $c : [0, D] \to X$ such that $c(0) = x$, $c(D) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, D]$. If $I \subseteq \mathbb{R}$ then a map $c : I \to X$ parametrizes its image proportional to arclength if there exists a constant $\lambda$ such that $d(c(t), c(t')) = \lambda|t - t'|$ for all $t, t' \in I$. We will often use geodesics parametrized proportional to arclength. Lastly, $(X, d)$ is a called a geodesic metric space if every pair of points are joined by a geodesic.

**Definition.** Let $(X, d)$ be a proper complete geodesic metric space. If $\triangle abc$ is a geodesic triangle in $X$, then we consider $\triangle \overline{abc}$ in $\mathbb{E}^2$, a triangle with the same side lengths, and call this a comparison triangle. Then we say $X$ satisfies the CAT(0) inequality if given $\triangle abc$ in $X$, then for any comparison triangle and any two points $p, q$ on $\triangle abc$, the corresponding points $\overline{p}, \overline{q}$ on the comparison triangle satisfy

$$ d(p, q) \leq d(\overline{p}, \overline{q}) $$

If $(X, d)$ is a CAT(0) space, then the following basic properties hold:

1. The distance function $d : X \times X \to \mathbb{R}$ is convex.
2. $X$ has unique geodesic segments between points.
3. $X$ is contractible.

For details, see [P].

Let $(X, d)$ be a proper CAT(0) space. First, define the boundary, $\partial X$ as a set as follows:

**Definition.** Two geodesic rays $c, c' : [0, \infty) \to X$ are said to be asymptotic if there exists a constant $K$ such that $d(c(t), c'(t)) \leq K, \forall t > 0$ - this is an equivalence relation. The boundary of $X$, denoted $\partial X$, is the set of equivalence classes of geodesic rays. The union $X \cup \partial X$ will be denoted $\overline{X}$. The equivalence class of a ray $c$ is denoted by $c(\infty)$.

There is a natural neighborhood basis for a point in $\partial X$. Let $c$ be a geodesic ray emanating from $x_0$ and $r > 0, \epsilon > 0$. Also, let $S(x_0, r)$ denote the sphere of radius $r$ centered at $x_0$ with $p_r : X \to B(x_0, r)$ denoting projection. Define

$$ U(c, r, \epsilon) = \{ x \in \overline{X} : d(x, x_0) > r, d(p_r(x), c(\infty)) < \epsilon \} $$
This consists of all points in $\overline{X}$ such that when projected back to $S(x_0, r)$, this projection is not more than $\varepsilon$ away from the intersection of that sphere with $c$. These set along with the metric balls about $x_0$ form a basis for the cone topology.

It is known that if $X_1$, $X_2$ are CAT(0) spaces, then $X_1 \times X_2$ is also CAT(0) and $\partial(X_1 \times X_2) \equiv \partial X_1 \ast \partial X_2$ where $\ast$ denotes the spherical join ([BH]). If $X = Y \times \mathbb{R}$, we obtain $\partial X \equiv \Sigma(\partial Y)$. We obtain the suspension as follows: Glue $\partial Y \times [0, \tfrac{\pi}{2}]$ and $\partial Y \times [-\tfrac{\pi}{2}, 0]$ together where the 0-levels of both copies are identified via the identity map on $\partial Y$ and the $\pm \tfrac{\pi}{2}$-levels in each are shrunk to a point. We denote the equivalence class of a point by $z = [z', \theta]$. With this notation, $\theta = 0$ means $z = z' \in \partial Y$ and $\theta = \pm \tfrac{\pi}{2}$ means $z$ is one of the suspension points.

Isometries of CAT(0) spaces can be divided into three types. This classification is based on the behavior of the displacement function for an isometry.

**Definition.** Let $\gamma$ be an isometry of the metric space $X$. The displacement function $d_\gamma : X \to \mathbb{R}_+$ defined by $d_\gamma(x) = d(\gamma \cdot x, x)$. The translation length of $\gamma$ is the number $|\gamma| = \inf \{ d_\gamma(x) : x \in X \}$. The set of points where $\gamma$ attains this infimum will be denoted $\text{Min}(\gamma)$. An isometry $\gamma$ is called semi-simple if $\text{Min}(\gamma)$ is non-empty.

We summarize some basic properties about this $\text{Min}(\gamma)$ in the following proposition, see [BH, chap 3].

**Proposition 1.1.** Let $X$ be a metric space and $\gamma$ an isometry of $X$.

1. $\text{Min}(\gamma)$ is $\gamma$-invariant.
2. If $\alpha \in \text{Isom}(X)$, then $|\gamma| = |\alpha \gamma \alpha^{-1}|$, and $\text{Min}(\alpha \gamma \alpha^{-1}) = \alpha \cdot \text{Min}(\gamma)$; in particular, if $\alpha$ commutes with $\gamma$, then it leaves $\text{Min}(\gamma)$ invariant.
3. If $X$ is CAT(0), then the displacement function $d_\gamma$ is convex: hence $\text{Min}(\gamma)$ is a closed convex subset of $X$.

The proofs of the first two properties are easy and the third follows directly from the fact that the distance function on $X$ is convex. Next we give the classification of isometries.

**Definition.** Let $X$ be a metric space. An isometry $\gamma$ of $X$ is called

1. elliptic if $\gamma$ has a fixed point - i.e $|\gamma| = 0$ and $\text{Min}(\gamma)$ is non-empty.
2. hyperbolic if $d_\gamma$ attains a strictly positive infimum.
3. parabolic if $d_\gamma$ does not attain its infimum, in other words if $\text{Min}(\gamma)$ is empty.

It is clear that an isometry is semi-simple if and only if it is elliptic or hyperbolic. If two isometries are conjugate in $\text{Isom}(X)$, then they are in the same class.

When a group $\Gamma$ acts geometrically on a CAT(0) space $X$, then the elements of $\Gamma$ act as semi-simple isometries because of the cocompactness of the action. Our main concern here will be the hyperbolic isometries. In [BH] there is a complete analysis of the three types of isometries, but we will not need the information about parabolics and elliptics here.

There is an important theorem about the structure of $\text{Min}(\gamma)$ when $\gamma$ is a hyperbolic isometry of a CAT(0) space $X$. The proof of this structure theorem relies on two theorems stated below. Both of these are proven in [BH] as well as the structure theorem. The first of these is the Flat Strip Theorem, a generalization of a theorem which holds in the theory of nonpositively curved manifolds, see [PCS].
Recall that a geodesic line in X is a map \( c: \mathbb{R} \to X \) such that \( d(c(t), c(t')) = |t - t'| \) for all \( t, t' \in \mathbb{R} \). Two such lines \( c, c' \) are asymptotic if there exists a constant \( K \) such that \( d(c(t), c'(t)) \leq K \) for all \( t \in \mathbb{R} \). Two lines are parallel if they cobound a flat strip. The following rigidity theorem shows that in a CAT(0) space, asymptotic lines are in fact parallel.

**The Flat Strip Theorem 1.2.** Let \( X \) be a CAT(0) space, and let \( c: \mathbb{R} \to X \) and \( c': \mathbb{R} \to X \) be geodesic lines in \( X \). If \( c \) and \( c' \) are asymptotic, then the convex hull of \( c(\mathbb{R}) \cup c'(\mathbb{R}) \) is isometric to a flat strip \( \mathbb{R} \times [0, D] \subseteq \mathbb{E}^2 \).

The next theorem provides a way of decomposing the set of parallel lines to a given line into a product. A proof can be found in [BH].

**The Decomposition Theorem 1.3.** Let \( X \) be a CAT(0) space and let \( c: \mathbb{R} \to X \) be a geodesic line in \( X \).

1. The union of the images of all geodesic lines \( c': \mathbb{R} \to X \) parallel to \( c \) is a convex subspace \( X_c \) of \( X \).

2. Let \( p \) be the restriction to \( X_c \) of the projection on the complete convex subspace \( c(\mathbb{R}) \). Let \( X^o_c = p^{-1}(c(0)) \). Then \( X^o_c \) is convex (in particular, it is also CAT(0)) and \( X_c \) is canonically isometric to the product \( X^o_c \times \mathbb{R} \).

The next theorem is the structure theorem for \( \text{Min}(\gamma) \) where \( \gamma \) is a hyperbolic isometry of a CAT(0) space \( X \). There are proofs available in [BH],[BR].

**Theorem 1.4.** Let \( X \) be a CAT(0) space.

1. An isometry \( \gamma \) of \( X \) is hyperbolic if and only if there exists a geodesic line \( c: \mathbb{R} \to X \) which is translated non-trivially by \( \gamma \), namely \( \gamma \cdot c(t) = c(t + a) \), for some \( a > 0 \). The set \( c(\mathbb{R}) \) is called an axis of \( \gamma \). For any such axis, the number \( a \) is equal to \( |\gamma| \).

2. If \( \gamma \) is hyperbolic, the axes of \( \gamma \) are all parallel to each other, and their union is \( \text{Min}(\gamma) \).

3. \( \text{Min}(\gamma) \) is isometric to a product \( Y \times \mathbb{R} \), and the restriction of \( \gamma \) to \( \text{Min}(\gamma) \) is of the form \( (y, t) \mapsto (y, t + |\gamma|) \), where \( y \in Y, \ t \in \mathbb{R} \).

4. Every isometry \( \alpha \) which commutes with \( \gamma \) leaves \( \text{Min}(\gamma) = Y \times \mathbb{R} \) invariant, and its restriction to \( Y \times \mathbb{R} \) is of the form \( (\gamma Y, \gamma t) \), where \( \gamma Y \) is an isometry of \( Y \) and \( \gamma t \) a translation of \( \mathbb{R} \).

The proof uses the Decomposition Theorem from above once (2) is established.

Another application of the Flat Strip Theorem is the following generalization of the above decomposition theorem for hyperbolic elements. It is called the Flat Torus Theorem and is proven in [BH] in the CAT(0) setting and [BGS] in the classical setting.

Recall that if \( A \) is an abelian group, then its rank \( r_{\mathbb{Q}} A \) is the dimension of the \( \mathbb{Q} \)-vector space \( A \otimes \mathbb{Q} \). In particular, if \( A \) is finitely generated then \( r_{\mathbb{Q}} A \) is the integer \( n \) such that \( A \) modulo its torsion subgroup is isomorphic to \( \mathbb{Z}^n \).

**Flat Torus Theorem 1.5.** Let \( \Gamma \) be a finitely generated abelian group acting properly by semi-simple isometries on a complete CAT(0) space \( X \).

1. \( \text{Min}(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{Min}(\gamma) \) is non-empty and splits as a product \( Y \times \mathbb{E}^n \), where \( n = r_{\mathbb{Q}} \Gamma \);

2. every \( \gamma \in \Gamma \) leaves \( \text{Min}(\Gamma) \) invariant, respecting the product structure; \( \gamma \) is the identity on the first factor \( Y \) and a translation on the second factor \( \mathbb{E}^n \).
(3) the quotient of each $n$-flat $\{y\} \times \mathbb{E}^n$ by this action is an $n$-torus.

**Corollary 1.6.** If $\Gamma$ is a group acting geometrically on a CAT(0) space $X$, and $H \cong \mathbb{Z}^n$ a subgroup of $\Gamma$, then there exists a point $x \in X$ such that the orbit $H \cdot x$ is a lattice in an isometrically embedded copy of $\mathbb{E}^n$ (i.e. an $n$-flat).

The next result shows that when we have an infinite order element in the center of a CAT(0) group $\Gamma$, there is a finite index subgroup of $\Gamma$ which contains this $Z$ as a direct factor. Using Proposition 1.8, we can then assume without loss of generality, that $\Gamma$ splits as $G \times \mathbb{Z}$. Recall that a subset $M$ of $X$ is quasi-dense if there is a constant $K \geq 0$ such that every point of $X$ lies within $K$ of $M$.

**Theorem 1.7 - [BH].** Let $X$ be a CAT(0) space and let $\Gamma$ be a finitely generated group acting by isometries on $X$. If $\Gamma$ contains a central subgroup $A \cong \mathbb{Z}^n$ that acts faithfully by hyperbolic isometries (apart from the identity element), then there exists a subgroup of finite index $H \subset \Gamma$ which contains $A$ as a direct factor.

**Proposition 1.8.** If $M \subset X$ is closed, convex, and quasi-dense, then $\partial X = \partial M$.

**Proof.** We know that $M$ is CAT(0) in the induced metric and $\partial M$ embeds in $\partial X$ so we need only show that this embedding is onto. This follows directly from the fact that if any geodesic ray in $X$ touches $M$, the image from that point on must be contained in $M$. For a proof of this fact, see Lemma 2.2.2 of [R]. If we choose our basepoint to be in $M$ all rays will have image in $M$ giving their endpoints in $\partial M$. □

We refer to the proof of the following theorem several times in this paper. This proof is given in [BR].

**Theorem 1.9.** Whenever $\Gamma = G \times \mathbb{Z}^n$ where $G$ is negatively curved acts geometrically on the CAT(0) space $X$, there is an embedding $\partial G \to \partial X$ that extends to a homeomorphism of the spherical join $\partial G \star S^{n-1}$ onto $\partial X$. Moreover, if $\Gamma$ also acts geometrically on the CAT(0) space $X'$, then there is a $\Gamma$-equivariant homeomorphism $\partial X \to \partial X'$; however, such a homeomorphism cannot in general be obtained as a continuous extension of a $\Gamma$-equivariant quasi-isometry of $X$ to $X'$.

Suppose we are in the $n = 1$ case of this theorem and suppose $\gamma$ generates the $Z$-factor of $\Gamma$. By Theorem 1.4 (3), $\text{Min}(\gamma)$ splits as $Y \times \mathbb{R}$ where $Y$ is a CAT(0) subset of $X$ in the induced metric.

In the proof of this theorem, we construct a geometric action of $G$ on $Y$ which we denote by $\ast$. We describe this action here since we refer to it in the proof of Theorem 4.2 and Lemma 4.3. We only consider the $n = 1$ case since that is the situation in this paper.

Again using Theorem 1.4 (4), each $(g, 0)$ leaves the product structure $Y \times \mathbb{R}$ invariant. In particular, $(g, 0)$ takes an axis for $\gamma$ to another axis for $\gamma$. Each $y \in Y$ lies on a unique axis $A_y$ for $\gamma$ and $(g, 0) \cdot A_y$ is another axis for $\gamma$ which intersects $Y$ in a unique point. We define $g \ast y$ to be this point, the unique point of $Y$ on the axis $(g, 0) \cdot A_y$.

**An Easy Example**

The group is $\Gamma = F_2 \times \mathbb{Z}$ acting two different ways on the CAT(0) space $X = T \times \mathbb{R}$ where $T$ denotes the tree of valence 4. $X$ is CAT(0); being the product of CAT(0) spaces, it is CAT(0).
of two CAT(0) spaces. Notice that $X$ is just the Cayley graph of $\Gamma$ with the appropriate generating set - that set being $\{(a^{\pm 1}, 0), (b^{\pm 1}, 0), (e, 1), (e, -1)\}$ where $a$ and $b$ generate $F_2$ as a free group with identity $e$ and $1$ generates $\mathbb{Z}$. Next, we describe the two actions. We let $(t, r)$ denote an element of $T \times \mathbb{R}$.

1. The first action is the standard product defined on the above generating set as follows:

   \[
   (a, 0) \cdot (t, r) = (a \cdot t, r) \\
   (b, 0) \cdot (t, r) = (b \cdot t, r) \\
   (e, 1) \cdot (t, r) = (t, r + 1)
   \]

   Where $a \cdot t, b \cdot t$ denote the usual action of $F_2$ on its Cayley graph $T$.

2. The second action is obtained by changing the action of $(b, 0)$ only.

   \[
   (a, 0) \ast (t, r) = (a \cdot t, r) \\
   (b, 0) \ast (t, r) = (b \cdot t, r + 2) \\
   (e, 1) \ast (t, r) = (t, r + 1)
   \]

Recall that a subset $Y$ of a geodesic metric space $X$ is called quasi-convex if there exists a $K \geq 0$ such that any geodesic in $X$ between two points of $Y$ lies in the $K$-neighborhood of $Y$.

In [BR] it is shown that the group $\Gamma = F_2 \times \mathbb{Z}$ acting with the two different actions described above on the CAT(0) space $X = T \times \mathbb{R}$ has the following property. We fix $x_0 = (e, 0)$, the root of the tree to be the basepoint:

The copy of $F_2$ given as the orbit $(F_2 \times \{0\}) \cdot x_0$ under the first action obviously lies in a horizontal slice of $T \times \mathbb{R}$ and is quasi-convex in $X$, however, the copy of $F_2$ given by $(F_2 \times \{0\}) \ast x_0$, the orbit of $F_2$ under the second action is NOT quasi-convex.

Using The Decomposition Theorem, it is clear that any CAT(0) space on which this $\Gamma$ acts geometrically splits as $Y \times \mathbb{R}$ with $Y$ quasi-isometric to $T$. One can use essentially the same argument as in [BR] to show the following claim.

**Claim 2.1. If $\Gamma$ acts geometrically on the CAT(0) space $X$ and the action is not the product action, then the $F_2$ factor is not quasi-convex.**

**Idea of Proof.** In [BR] we construct a sequence of group elements $\{g_i\}$ such that the midpoint of the geodesic segment from $x_0$ to $x_i = (g_i, 0) \ast x_0$ (where we are using the second action from above) lies further than $\frac{1}{4}$ units from the orbit $(F_2 \times \{0\}) \ast x_0$. In fact, $g_i = a^ib^i$ and the pertinent point here is that the $b$ generator has vertical translation in the direction of the generator for $\mathbb{Z}$. One can adjust generating sets and the selection of the $g_i$ to make this argument work in the general case. □

**Angles in CAT(0) Spaces**

In the above example, we saw that when a group of the form $\Gamma = G \times \mathbb{Z}$ acts geometrically on a CAT(0) space $X$, the subgroup $G \times \{0\}$ does not generally determine a convex (or even quasi-convex) subset of $X$ under the induced action as one might expect. A next question might be: How badly CAN the orbit of $G \times \{0\}$ sit inside $X$? This action will provide a bit of an answer to that question.
Specifically, we will show that there is a well-defined “angle” between the orbits of $G \times \{0\}$ and $\{e\} \times \mathbb{Z}$. As a consequence of this, the limit set of the orbit of $G \times \{0\}$ will not contain the endpoints of an axis for the $\mathbb{Z}$-factor - i.e. the suspension points in $\partial X = \Sigma(\partial Y)$.

In the above example, the angle that $F_2$ makes with the $\mathbb{Z}$ factor is exactly $\frac{\pi}{2} - \arctan(2)$ - this will be clear from the proof of the main theorem.

In this section, we develop a general technique for measuring angles between points in $\partial X$. We assume the reader has some knowledge of how to measure angles in a metric space, but we add the necessary definitions for completeness. Alexandrov used the method of comparison triangles to define the notion of angle between two geodesics leaving a point $x_0$ in a metric space $X$, [A]. We recall that definition here.

**Definition.** Let $c : [0,a] \to X$ and $c' : [0,a'] \to X$ be two geodesics with $c(0) = c'(0) = x_0$. Given $t \in [0,a], t' \in [0,a']$, and let $\alpha_{t,t'}$ denote the angle in a comparison triangle in Euclidean space at the vertex corresponding to $x_0$. The (upper) angle between $c, c'$ at $x_0$ is defined to be the following number:

$$\angle_{c, c'} := \limsup_{t,t' \to 0} \alpha_{t,t'}$$

Note: The limsup is used because the limit may not always exist, but in CAT(0) spaces, the limit does exist and instead of calling it an “upper” angle, we call it the angle. For a proof of this, we reference [BH].

**Definition.** Let $X$ be a CAT(0) space. Given $x \in X$ and $u, v \in \partial X$, we denote by $\angle_x(u, v)$ the angle between the unique geodesic rays which issue from $x$ and lie in the classes $u$ and $v$ respectively. Then we define the angle between $u$ and $v$ to be

$$\angle(u, v) = \sup_{x \in X} \angle_x(u, v)$$

The following proposition gives some basic facts about angles between boundary points of a CAT(0) space (need completeness here which is assumed for all spaces here). The proofs of these involve repeated use of the CAT(0) inequality and the convexity of the distance function. For complete proofs, see chapter 3 of [BH].

**Proposition 3.1.** Let $X$ be a CAT(0) space and let $c, c'$ be two geodesic rays issuing from the same point $x \in X$. Let $u = c(\infty)$ and $u' = c'(\infty)$.

1. If $u \neq u'$, then there exists $t > 0$ such that $\angle_{c(t)}(u, u') > 0$; hence $\angle(u, u') > 0$.
2. The function $y \mapsto \angle_{y}(u, u')$ is upper semicontinuous on $X$.
3. The function $t \mapsto \angle_{c(t)}(u, u')$ is non-decreasing and

$$\angle(u, u') = \lim_{t \to \infty} \angle_{c(t)}(u, u')$$

4. If $\angle_x(u, u') = \angle(u, u')$, then the convex hull of $c(\mathbb{R}_+) \cup c(\mathbb{R}_+)$ is isometric to a sector in the Euclidean plane bounded by two rays which meet at an angle $\angle(u, u')$. 


(5) Suppose \( \{x_n\}, \{y_n\} \) are sequences of points of \( X \) converging to \( u, u' \) in \( \partial X \). Then if \( x_0 \) is a basepoint of \( X \),

\[
\liminf_{n \to \infty} \angle_{x_0}(x_n, y_n) \geq \angle(u, u')
\]

where \( \angle_{x_0}(x_n, y_n) \) denotes the angle in a Euclidean comparison triangle.

It is easy to see that this angle satisfies the triangle inequality and so this combined with (1) from above gives us that \( \angle(u, u') \) defines a metric on \( \partial X \) called the angular metric. The next proposition provides an asymptotic formula for computing the angle between points in the boundary. The proof is easy and is left as an exercise.

**Proposition 3.2.** Let \( x \) be a point in the \( \text{CAT}(0) \) space \( X \) and let \( c, c' \) be two geodesic rays emanating from \( x \) with \( u = c(\infty) \) and \( u' = c'(\infty) \). Then

\[
\lim_{t \to \infty} \frac{d(c(t), c'(t))}{t} = [2 - 2 \cos(\angle(u, u'))]^\frac{1}{2}
\]

We will need the following proposition for computing the angle between rational endpoints of infinite order elements. The proof uses the fact that given a ray \( c \) and a point \( y \) not in the image of \( c \), there exists a unique ray \( c' \) beginning at \( y \) asymptotic to \( c \), see chapter 3 of [BH] for details.

**Proposition 3.3.** The expression on the left hand side of the asymptotic formula in Proposition 3.2 is independent of basepoint.

**Proof.** Let \( y \) be another point in \( X \). Let \( r, r' \) denote rays beginning at \( y \) which are asymptotic to \( c \) and \( c' \) respectively. Now we have,

\[
d(r(t), r'(t)) \leq (d(r(t), c(t)) + d(c(t), c'(t)) + d(c'(t), r'(t)) \leq 2d(x, y) + d(c(t), c'(t))
\]

The first follows from the triangle inequality, the second follows from the convexity of the distance function - indeed, consider the quadrilateral formed with vertices \( x, y, r(t), c(t) \), we know the side formed by \( r(t) \) and \( c(t) \) is no bigger than the side formed by \( x \) and \( y \). Likewise for the pair \( r'(t), c'(t) \). Dividing by \( t \) and taking the limit as \( t \to \infty \) gives the independence of basepoint as \( \frac{2d(x, y)}{t} \to 0 \) as \( t \to \infty \). \( \square \)

**Example.** Consider the \( \text{CAT}(0) \) space \( X = \mathbb{H}^2 \). We know \( \partial X \) is the unit circle. For any two points on the boundary circle, there is a geodesic line in \( \mathbb{H}^2 \) joining these two points. Because the above calculation is independent of basepoint, we may as well assume the basepoint is on this line. Then one easily sees that the angle between the two points must be \( \pi \). This means the angle metric on \( \partial(\mathbb{H}^2) \) is discrete.

**The Angle Question**

For this section, let \( \Gamma \) be a group acting geometrically on the \( \text{CAT}(0) \) space \( X \) and suppose \( \gamma \) is an infinite order element in the center. By Theorem 1.7, there exists a finite index subgroup of the form \( G \times \mathbb{Z} \). Since we are considering questions concerning \( \partial X \), we can take \( \Gamma = G \times \mathbb{Z} \). Even though the original motivation for this problem involved the assumption that \( G \) is word hyperbolic, the proofs in this
section go through without it. Indeed, we know $\text{Min}(\gamma) = Y \times \mathbb{R}$ and the proof of Theorem 1.9 provides an action of $G$ on the $Y$ factor which is geometric. The fact that this action is geometric does not use the assumption of $G$ being word hyperbolic. In fact, assuming $\Gamma$ is a CAT(0) group implies $G$ is a CAT(0) group because of this action.

In the case where $G$ is word hyperbolic we get a somewhat nicer statement since the concept of boundary for these groups is well-defined and it is known that the rational points are dense in $\partial G$. See the remark after Theorem 4.5 for details.

Recall for each $g \in G$, we denote by $g^\infty$, the rational endpoint of $\partial X$ determined by the element $(g,0)$ of $\Gamma$. This endpoint is

$$g^\infty = \lim_{i \to \infty} (g^i,0) \cdot x$$

for some (and hence any) $x \in X$, where this limit is taken in the cone topology on $X \cup \partial X$. We observe the following lemma:

**Lemma 4.1.** For $g \in G$, $g^\infty \neq \gamma^\infty$. In fact, $\{g^{\pm \infty}\} \cap \{\gamma^{\pm \infty}\} = \emptyset$.

**Proof.** We know that the subgroup of $\Gamma$ generated by $(g,0)$ and $\gamma$ is a copy of $\mathbb{Z} \oplus \mathbb{Z}$ which acts by semisimple isometries on $X$. Thus we can apply the Flat Torus Theorem to obtain a point $x \in \text{Min}(\langle (g,0) \rangle) \cap \text{Min}(\gamma)$. Since $x \in \text{Min}(\langle (g,0) \rangle)$, we know that the convex hull of the orbit $\langle (g,0) \rangle \cdot x$ is an axis for $(g,0)$, denoted $A_g$ and likewise, the convex hull of the orbit $\langle \gamma \rangle \cdot x$ lies along an axis $A_\gamma$. Denote by $A_g^+$, the part of $A_g$ beginning at $x$ and ending at $g^\infty \in \partial X$. Likewise for $A_\gamma^+$, if $g^\infty = \gamma^\infty$, then we have $A_g^+ = A_\gamma^+$ by uniqueness of geodesics. Also, we know $(g,0)$ leaves $A_g$ fixed as a set and takes $A_\gamma$ to another axis of $\gamma$. Thus $(g,0) \cdot A_\gamma$ is a geodesic line through $x$ which coincides with $A_\gamma^+ = A_g^+$ and is also parallel to $A_\gamma$. Thus we must have $(g,0) \cdot A_\gamma = A_\gamma$ - i.e. two parallel lines which intersect, must be the same line. Thus since both $(g,0)$ and $\gamma$ act as translation along any axis, we would have that the entire orbit of $x$ under $\langle (g,0), \gamma \rangle$ lies along this line. But the Flat Torus Theorem tells us that the convex hull of this orbit must be an entire 2-plane. Thus we must have $g^\infty \neq \gamma^\infty$. The last statement of the theorem now follows easily as we can replace $g^\infty$ with $g^{-\infty}$ and likewise for $\gamma$ in the above argument. $\Box$

Consider the following:

$$\Theta(G) = \inf\{\angle(g,\gamma) : g \in G, \; o(g) = \infty\}$$

where $\angle(g,\gamma)$ means the angle between the rational endpoints in $\partial X$ for the elements $(g,0)$ and $\gamma$. The theorem below will show that this “angle” of $G$ is bounded away from zero. Using the notation in the above proof, we now outline a method for calculating $\angle(g,\gamma)$. From the Flat Torus Thm, there exists a point $x_0 \in \text{Min}(\langle (g,0) \rangle) \cap \text{Min}(\gamma)$ and the convex hull of the orbit $\langle (g,0), \gamma \rangle \cdot x_0$ is a Euclidean plane. This plane lies on an axis for $(g,0)$ and also an axis for $\gamma$. We can view this point as the origin in this 2-plane and view the axis for $\gamma$ as the “y-axis” for intuition purposes. By Lemma 4.1, we know the axis for $(g,0)$ must be a non-vertical line in this plane. In particular, the axis $A_g$ either determines a horizontal line or a line with positive or negative slope given this vertical line for reference. Without loss of generality, suppose that $A_\gamma^+$ lies in the first quadrant (i.e. the line
has positive slope). We will calculate the angle \( \angle (g, \gamma) \) using Proposition 3.2. Since this plane is isometrically embedded in \( X \), calculations involving distances in the plane are the same as distances measured in \( X \).

Within this Euclidean plane, consider the rays \( A^+_g \) and \( A^+_g \) emanating from the “origin” \( x_0 \) and ending at \( \gamma^\infty \) and \( g^\infty \) respectively. If we let \( d(t) = d(A^+_g (t), A^+_g (t)) \) and let \( \theta_g \) denote the angle between these rays at \( x_0 \) - i.e. we know how to calculate angles in Euclidean space. For each \( t \), consider the isosceles triangle with vertices \( x_0, A^+_g (t), A^+_g (t) \). Using elementary trigonometry, we have \( \sin \left( \frac{\theta_g}{2} \right) = \frac{d(t)}{t} \) which gives

\[
\frac{d(t)}{t} = 2 \sin \left( \frac{\theta_g}{2} \right) = \left[ 2 - 2 \cos \theta_g \right]^{\frac{1}{2}}
\]

This calculation is independent of \( t \).

Finally, applying the formula in Proposition 3.2 - and using the fact that this calculation is independent of basepoint chosen - we get:

\[
2 \sin \left( \frac{\angle (g, \gamma)}{2} \right) = \lim_{t \to \infty} \frac{d(t)}{t} = 2 \sin \left( \frac{\theta_g}{2} \right)
\]

Since we can assume \( 0 < \angle (g, \gamma) < \frac{\pi}{2} \) by passing to inverses if necessary and the sine function is 1-1 on \( [0, \frac{\pi}{2}] \), we have \( \angle (g, \gamma) = \theta_g \).

Recall that \( \text{Min}(\gamma) \) decomposes as \( Y \times \mathbb{R} \) and that for each \( g \in G \), \( (g, 0) \) acts on \( \text{Min}(\gamma) \) via \( (g_Y, \tau_g) \) where \( g_Y \in \text{Isom}(Y) \) and \( \tau_g \) is a translation since \( (g, 0) \) commutes with \( \gamma \) (see Proposition 1.1). The plane used above actually sits inside \( \text{Min}(\gamma) \cap \text{Min}((g, 0)) \). Consider the point \( (g, 0) \cdot x_0 \) inside this plane (notice, we can assume \( x_0 \in Y \)). This point has “coordinates” \( (|g|, \tau_g) \) where \( |g| = d(x_0, g \ast x_0) \) where \( \ast \) denotes the action of \( G \) on \( Y \) described after the statement of Theorem 1.9. This gives a formula for the slope of \( A^+_g \) inside this plane, namely:

\[
s(g) = \frac{\tau_g}{|g|}
\]

The geometry now gives:

\[
\angle (g, \gamma) = \theta_g = \frac{\pi}{2} - \tan^{-1}(s(g))
\]

We are now ready to prove the following:

**Theorem 4.2.** Let \( \Theta(G) = \inf \{ \angle (g, \gamma) : g \in G, o(g) = \infty \} \) be the “angle” of \( G \) inside \( X \). Then \( \Theta(G) > 0 \) - in particular, no rational ray can limit to a suspension point.

First we prove two necessary lemmas.

**Lemma 4.3.** For the \( \ast \) action of \( G \) on \( Y \) described after the statement of Theorem 1.9, \( |g| = d(x_0, g \ast x_0) \) where \( x_0 \in \text{Min}((g, 0)) \cap Y \) for the action of \( \Gamma \) on \( X \) gives the minimum displacement of \( g \) on \( Y \) - thus the chosen notation is unambiguous.

**Proof.** This follows easily from the construction of the \( \ast \) action and the fact that \( x_0 \in \text{Min}((g, 0)) \). Indeed, \( (g, 0) \) takes the axis of \( \gamma \) containing \( x_0 \) to the axis of \( \gamma \) containing \( (g, 0) \cdot x_0 \) and the distance between these two parallel lines is \( ||(g, 0)|| - \) the minimum displacement for the element \( (g, 0) \) of \( \Gamma \) acting on \( X \). Thus there can be no closer parallel line, and the result follows. \( \square \)

The following is a known result and can be found in [RH] [G].
Lemma 4.4. Suppose a group $\Gamma$ acts geometrically on a CAT(0) space $X$. Then there exists a $K > 0$ such that for all $\gamma \in \Gamma$ with $\gamma$ hyperbolic, $|\gamma| \geq K$.

Proof - Theorem 4.2. From above we have,

$$\angle(g, \gamma) = \theta_g = \frac{\pi}{2} - \tan^{-1}(s(g))$$

To prove the theorem, it suffices to squeeze $s(g)$ between two finite numbers independent of $g \in G$ to insure that $\angle(g, \gamma)$ is bounded away from 0. Let $S$ be a finite (symmetric) generating set for the $G$-factor of $\Gamma = G \times \mathbb{Z}$. Let $M = \max \{|\tau_s| : s \in S\}$ where $\tau$ comes from considering the action of $\Gamma$ on $\text{Min}(\gamma) = Y \times \mathbb{R}$. Let $K > 0$ be as in Lemma 4.4 where we consider the geometric action of $G$ on $Y$. Since this is a geometric action, there is $(\lambda, \epsilon)$ quasi-isometry between $G$ and $Y$ where $G$ is given the word metric associated to the generating set $S$. We denote by $l_S(g)$ the word length of the element $g \in G$. Recall, this means for all $g \in G$, and for all $y \in Y$ - in particular for $x_0$ as above:

$$\frac{1}{\lambda}l_S(g) - \epsilon \leq |g| \leq \lambda l_S(g) + \epsilon$$

First assume $s(g) = \frac{\tau_g}{|g|} \geq 0$, otherwise use $g^{-1}$ in place of $g$. Also, if $g = s_1s_2 \cdots s_n$ is a shortest representative for $g$ in terms of $S$, then it is clear that $\tau_g = \sum_{i=1}^n \tau_{s_i}$, since these are just translations of the $\mathbb{R}$-factor.

First consider the case where the word length of $g$ is small - say $1 \leq l_S(g) \leq D = \lambda \epsilon - 1$. In this case, we have:

$$0 \leq s(g) = \frac{\tau_g}{|g|} = \frac{\sum_{i=1}^n \tau_{s_i}}{|g|} \leq \frac{M \cdot l_S(g)}{K} \leq \frac{M \cdot D}{K}$$

Next, consider the case when $l_S(g)$ is large, we have:

$$0 \leq s(g) = \frac{\tau_g}{|g|} = \frac{\sum_{i=1}^n \tau_{s_i}}{|g|} \leq \frac{M \cdot l_S(g)}{K} \leq \frac{M}{\lambda l_S(g) - \epsilon} \leq \frac{M}{\lambda - \frac{\epsilon}{l_S(g)}}$$

the last inequality holding true since we are assuming $l_S(g) > D$. In either case, we have $0 \leq s(g) \leq N$ where $N = \max\{\frac{M \cdot D}{K}, \frac{M}{\lambda - \frac{\epsilon}{D}}\}$. □

Now we wish to show something a bit stronger about how the $G$-factor sits inside $X$ by considering the limit set of $(G \times \{0\}) \cdot x_0$ ($x_0 \in Y$). Let $\mathcal{L}(G)$ denote this limit set.

A point $z \in \partial X$ is in $\mathcal{L}(G)$ iff there exists $\{g_n\} \subseteq G$ such that $\lim_{n \to \infty} (g_n, 0) \cdot x_0 = z$ where the limit is taken in the cone topology on $X \cup \partial X$. We want to measure the angle $\angle(z, \gamma^\infty)$ to see how this limit set sits inside $X$. We know there is an axis for $\gamma$ beginning at $x_0$ and ending at $\gamma^\infty$. Trivially then, if we let $x_n$ be the point at distance $n$ away from $x_0$ along this axis, then $\lim_{n \to \infty} x_n = \gamma^\infty$.

Theorem 4.5. There exists a constant $0 \leq B < \frac{\pi}{2}$ such that $\mathcal{L}(G)$ is contained in the $[-B, B]$ interval of the suspension $\partial X = \Sigma(\partial Y)$. Explicitly, for $z \in \mathcal{L}(G)$, $z = [z', \theta] \in \Sigma(\partial Y)$ where $z' \in \partial Y$ and $|\theta| \leq B$.

Proof. To see this, we measure the angle $\angle(\gamma^\infty, z)$. We will use the sequences $\mathcal{x_n} \to \gamma^\infty$ (from above) and $y_n = (g_n, 0) \cdot x_0 \to z$ and Theorem 3.1 (5). The important observation here is:

$$\overline{x_n} \in \angle(\gamma^\infty, y_n)$$
This is true because we know there is a basepoint $p \in X$ for which the rays from $p$ to $\gamma^\infty$ and $g_n^\infty$ form a Euclidean sector - i.e. any comparison angle will have to be this Euclidean angle. Now the result follows easily from Theorem 3.1 (5) and Theorem 4.2. Explicitly, from Theorem 4.2, there exists $\varepsilon > 0$ such that for all $n$, 
$$\angle(\gamma^\infty, g_n^\infty) \geq \varepsilon$$ - i.e. these are all bounded away from zero. We now have,

$$\varepsilon \geq \lim \inf_{n \to \infty} \angle(\gamma^\infty, g_n^\infty) = \lim \inf_{n \to \infty} \angle x_0(x_n, y_n) \geq \angle(\gamma^\infty, z)$$

If $z = [z', \theta]$, then by our notation, $\theta = \frac{\pi}{2} - \angle(\gamma^\infty, z)$. So we can take $B = \frac{\pi}{2} - \varepsilon$.

$\square$.

\textbf{Remark.} In the case where $G$ is word hyperbolic, we know $\partial Y$ is homeomorphic to $\partial G$, the Gromov boundary of the group so that $\partial X \equiv \Sigma(\partial G)$. Also, the rational points are dense in $\partial G$ in this case, so one might expect that Theorem 4.5 is redundant here - but it is not since the map on rational points does not extend to a continuous map of $\partial G$ into $\partial X$. This is shown precisely in Theorem 1.9.

\textbf{REFERENCES}


\textbf{Department of Mathematics Vanderbilt University Nashville, TN 37240}

\textit{E-mail address: ruane@math.vanderbilt.edu}