The determinant of a triangular matrix is the sum of the elements on the main diagonal.
\[ u = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \]
Compute the area of the parallelogram determined by \( \mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}, \) and \( \mathbf{0} \), and compute determinant of \([ \mathbf{u} \ \mathbf{v} ]\). How do they compare? Replace first entry of \( \mathbf{v} \) by an arbitrary number \( x \), and repeat the item. Draw a picture and explain what you find.
\[ \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}, \text{ where } a, b, c \text{ are positive (for simplicity). Compute the area of the parallelogram determined by } \mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}, \text{ and } \mathbf{0}, \text{ and compute the determinants of matrices } [ \mathbf{u} \ \mathbf{v} ] \text{ and } [ \mathbf{v} \ \mathbf{u} ] \text{. Draw a picture and explain what you find.} \]

Is it true that \( \det( \mathbf{A} + \mathbf{B} ) = \det( \mathbf{A} ) + \det( \mathbf{B} )? \) To find random matrices \( 5 \times 5 \) matrices \( \mathbf{A} \) and \( \mathbf{B} \), and compute \( \det( \mathbf{A} + \mathbf{B} ) = \det( \mathbf{A} ) - \det( \mathbf{B} ). \) (Refer to Exercise 37 in Section 2.1.) Repeat the calculations for three other pairs of \( n \times n \) matrices, for various values of \( n \). Report your results.

44. [M] Is it true that \( \det( \mathbf{A} \mathbf{B} ) = (\det( \mathbf{A} ))(\det( \mathbf{B} ))? \) Experiment with four pairs of random matrices as in Exercise 43, and make a conjecture.

45. [M] Construct a random \( 4 \times 4 \) matrix \( \mathbf{A} \) with integer entries between \(-9\) and \(9\), and compare \( \det( \mathbf{A} ) \) with \( \det( \mathbf{A}^T ) \), \( \det( -\mathbf{A} ) \), \( \det(2\mathbf{A} ) \), and \( \det(10\mathbf{A} ) \). Repeat with two other random \( 4 \times 4 \) integer matrices, and make conjectures about how these determinants are related. (Refer to Exercise 36 in Section 2.1.) Then check your conjectures with several random \( 5 \times 5 \) and \( 6 \times 6 \) integer matrices. Modify your conjectures, if necessary, and report your results.

46. [M] How is \( \det( \mathbf{A}^{-1} ) \) related to \( \det( \mathbf{A} )? \) Experiment with random \( n \times n \) integer matrices for \( n = 4, 5, \) and \( 6 \), and make a conjecture. Note: In the unlikely event that you encounter a matrix with a zero determinant, reduce it to echelon form and discuss what you find.

**SOLUTION TO PRACTICE PROBLEM**

Take advantage of the zeros. Begin with a cofactor expansion down the third column to obtain a \( 3 \times 3 \) matrix, which may be evaluated by an expansion down its first column.
\[
\begin{vmatrix}
5 & -7 & 2 & 2 \\
0 & 3 & 0 & -4 \\
-5 & -8 & 0 & 3 \\
0 & 5 & 0 & -6
\end{vmatrix} = (-1)^{1+3} \begin{vmatrix}
0 & 3 & -4 \\
-5 & -8 & 3 \\
0 & 5 & -6
\end{vmatrix} = 2 \cdot (-1)^{2+1}(-5) \begin{vmatrix}
3 & -4 \\
5 & -6
\end{vmatrix} = 20
\]
The \((-1)^{2+1}\) in the next-to-last calculation came from the \((2, 1)\)-position of the \(-5\) in the \(3 \times 3\) determinant.

**PROPERTIES OF DETERMINANTS**

The secret of determinants lies in how they change when row operations are performed. The following theorem generalizes the results of Exercises 19-24 in Section 3.1. The proof is at the end of this section.

**THEOREM 3**

Row Operations

Let \( \mathbf{A} \) be a square matrix.

a. If a multiple of one row of \( \mathbf{A} \) is added to another row to produce a matrix \( \mathbf{B} \), then \( \det( \mathbf{B} ) = \det( \mathbf{A} ) \).

b. If two rows of \( \mathbf{A} \) are interchanged to produce \( \mathbf{B} \), then \( \det( \mathbf{B} ) = -\det( \mathbf{A} ) \).

c. If one row of \( \mathbf{A} \) is multiplied by \( k \) to produce \( \mathbf{B} \), then \( \det( \mathbf{B} ) = k \cdot \det( \mathbf{A} ) \).

The following examples show how to use Theorem 3 to find determinants efficiently.
EXAMPLE 2. Compute \( \det A \), where \( A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} \).

**Solution**: The strategy is to reduce \( A \) to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

\[
\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \end{vmatrix} = 1 \cdot (-5) = -5
\]

An interchange of rows 2 and 3 reverses the sign of the determinant, so

\[
\det A = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15
\]

A common use of Theorem 3(c) in hand calculations is to factor out a common multiple of one row of a matrix. For instance,

\[
\begin{vmatrix} * & * & * \\ 5k & -2k & 3k \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ 5 & -2 & 3 \\ * & * & * \end{vmatrix}
\]

where the starred entries are unchanged. We use this step in the next example.

EXAMPLE 2. Compute \( \det A \), where \( A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} \).

**Solution**: To simplify the arithmetic, we want a 1 in the upper-left corner. We can interchange rows 1 and 4. Instead, we factor out 2 from the top row, and then proceed with row replacements in the first column:

\[
\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}
\]

Next, we could factor out another 2 from row 3 or use the 3 in the second column pivot. We choose the latter operation, adding 4 times row 2 to row 3:

\[
\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}
\]

Finally, adding \(-1/2\) times row 3 to row 4, and computing the "triangular" determinant, we find that

\[
\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \cdot (1)(3)(-6)(1) = -36
\]
Suppose a square matrix \( A \) has been reduced to an echelon form \( U \) by row replacements and row interchanges. (This is always possible. See the row reduction algorithm in Section 1.2.) If there are \( r \) interchanges, then Theorem 3 shows that

\[
\det A = (-1)^r \det U
\]

Since \( U \) is in echelon form, it is triangular, and so \( \det U \) is the product of the diagonal entries \( u_{11}, \ldots, u_{nn} \). If \( A \) is invertible, the entries \( u_{ii} \) are all pivots (because \( A \sim I_n \) and the \( u_{ii} \) have not been scaled to 1’s). Otherwise, at least \( u_{nn} \) is zero, and the product \( u_{11} \cdots u_{nn} \) is zero. See Fig. 1. Thus

\[
det A = \begin{cases} 
(-1)^r \cdot \text{(product of pivots in } U) & \text{when } A \text{ is invertible} \\
0 & \text{when } A \text{ is not invertible}
\end{cases}
\tag{1}
\]

It is interesting to note that although the echelon form \( U \) described above is not unique (because it is not completely row reduced), and the pivots are not unique, the product of the pivots is unique, except for a possible minus sign.

Formula (1) not only gives a concrete interpretation of \( \det A \) but also proves the main theorem of this section:

**Theorem 4**

A square matrix \( A \) is invertible if and only if \( \det A \neq 0 \).

Theorem 4 adds the statement "\( \det A \neq 0 \)" to the Invertible Matrix Theorem. A useful corollary is that \( \det A = 0 \) when the columns of \( A \) are linearly dependent. Also, \( \det A = 0 \) when the rows of \( A \) are linearly dependent. (Rows of \( A \) are columns of \( A^T \), and linearly dependent columns of \( A^T \) make \( A^T \) singular. When \( A^T \) is singular, so is \( A \), by the Invertible Matrix Theorem.) In practice, linear dependence is obvious when two columns or two rows are the same or a column or a row is zero.

**Example 3** Compute \( \det A \), where \( A = \begin{bmatrix}
3 & -1 & 2 & -5 \\
0 & 5 & -3 & -6 \\
-6 & 7 & -7 & 4 \\
-5 & -8 & 0 & 9
\end{bmatrix} \)

**Solution** Add 2 times row 1 to row 3 to obtain

\[
\det A = \det \begin{bmatrix}
3 & -1 & 2 & -5 \\
0 & 5 & -3 & -6 \\
0 & 5 & -3 & -6 \\
-5 & -8 & 0 & 9
\end{bmatrix} = 0
\]

because the second and third rows of the second matrix are equal.

1. Most computer programs that compute \( \det A \) for a general matrix \( A \) use the method of formula (1) above.
2. It can be shown that evaluation of an \( n \times n \) determinant using row operations requires about \( 2n^3/3 \) arithmetic operations. Any modern microcomputer can calculate a \( 25 \times 25 \) determinant in a fraction of a second, since only about 10,000 operations are required.