Math 150-02
Homework 1
Spring 2013

Instructions: You may work together on this, but if you do, YOU MUST tell me with whom you worked, and on which problems. Furthermore, all answers must be written in your own words. This is due back the beginning of class, Wed. Jan. 30, 2013.

1. An $n \times n$ Toeplitz matrix is one that is constant along its diagonals. For example,

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 3 & 4 & -2 \\ 0 & 3 & 4 \end{bmatrix}$$

is an example of a $3 \times 3$ Toeplitz matrix.

Fix $n$ to be a positive integer, $n \geq 2$. Consider the set of all $n \times n$ Toeplitz matrices with real-valued entries, which we’ll denote $T_{n \times n}$ for lack of something better. Show that this is a subspace of $\mathbb{R}^{n \times n}$.

2. Show that the following 5 elements are a basis for $T_{3 \times 3}$ (we will call these the standard basis elements for this space):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. Let $v \in \mathbb{R}^{5 \times 1}$. Consider the map that takes $v$ to a Toeplitz matrix in $\mathbb{R}^{3 \times 3}$ according to the following rule:

$$\mathcal{L}(v) = \mathcal{L}\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_5 \end{bmatrix}\right) = \begin{bmatrix} v_3 & v_2 & v_1 \\ v_4 & v_3 & v_2 \\ v_5 & v_4 & v_3 \end{bmatrix}.$$

Answer the following.

(a) Show that $\mathcal{L}$ is a linear transformation from $\mathbb{R}^5$ to $T_{3 \times 3}$.

(b) Show that $\mathbb{R}^5$ is isomorphic to $T_{3 \times 3}$.

4. Find the matrix of the transformation for $\mathcal{L}$ defined in the previous problem, where the basis we will use for $\mathbb{R}^5$ is

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, b_4 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, b_5 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}, b_6 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$ 

and the basis to use for $T_{3 \times 3}$ is the standard basis in question 2.

5. Let $\{v_1, v_2\}$ be a linearly independent set in a vector space $V$, and define $V_1 = \text{span}(v_1, v_2)$. Let $\{w_1, w_2\}$ be another linearly independent set in $V$ and define $W_1 = \text{span}(w_1, w_2)$.

(a) Suppose you know nothing else about $V_1$ and $W_1$. Consider the subspace $V_1 + W_1$. Give the maximum possible dimension of this space and the minimum possible dimension of this space, and explain your reasoning.

(b) Now suppose you know $\dim(V_1 + W_1) = 3$. Am I guaranteed that a basis for the subspace $V_1 + W_1$ is given by either $\{v_1, v_2, w_1\}$ or $\{v_1, v_2, w_2\}$? Why or why not?

(c) Now suppose you know $\dim(V) = 4$, and $\dim(V_1 + W_1) = 4$. Show that in this case $V = V_1 \oplus W_1$.

6. Show that, for $f(t) = 1$, $\{e^t, e^{-t}, f(t)\}$ are linearly independent functions on $C[-2,2]$. You can argue this directly from the definition, or you can use determinants in a manner similar to the ODE example in the notes.