Math 150-02  
Take-home Exam 2  
Spring 2013

Instructions: The ONLY MATERIALs you may use are a) anything on the 150-02 course webpage, including answer keys for homeworks b) notes you took in class c) your graded homeworks. You may NOT use any other textbooks or notes. You may not consult any other online sources or use any computer programs. You may not discuss the exam with anyone but the instructor. You are bound by Tufts honor code, so please be sure to sign your name somewhere on the exam solutions - your signature signifies your understanding of these rules. The exam is DUE the beginning of class, Wed., April. 17, 2013.

1. Let $A$ be a $5 \times 5$ matrix. Suppose I told you that the minimal degree polynomial for $A$ is of degree 3. What is the maximum number of distinct eigenvalues that $A$ has? The minimum? Justify.

2. Consider $V = \mathbb{C}^{2 \times 2}$.

   $f(A) = \max(|\lambda_1|, |\lambda_2|)$.

   (a) Let $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a, b \in \mathbb{C} \right\}$. It’s easy to show that $H$ is a subspace of $V$ (and therefore is itself a vector space) - you don’t have to do this. What I want to know is whether or not $f$ defines a valid norm on $H$? If yes, show it, if no, explain why not.

   (b) Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in \mathbb{C} \right\}$. It’s easy to show that $H$ is a subspace of $V$ (and therefore it is itself a vector space). Does $f(A)$ define a norm over this space? If yes, show it, if no, explain why not.

3. Let $V$ be a vector space of finite dimension $n$, and let $H = \text{span}\{q_1, \ldots, q_k\}$, where $q_1, \ldots, q_k$ are orthonormal in a given inner product for $V$. I’ve said orthonormal vectors are linearly independent (you can assume it here without proof, though the proof is about 3 lines). Let $\mathcal{L}_H$ denote the orthogonal projector onto $H$. Answer the following.

   (a) Show $\mathcal{L}_H$ is linear. (Technically, it’s a linear operator since its range resides also in $V$).

   (b) Recall (chapt 1) any set of $k$ linearly independent vectors in $V$ can be extended to a basis for $V$. So let $\{q_1, \ldots, q_k, b_{k+1}, \ldots, b_n\}$ be such a basis for $V$. By Gram-Schmidt, we can find an orthogonal basis for $V$ using this basis as the starting point – of course, part of the work in the G-S process is already done because the first $k$ vectors in this basis for $V$ are already orthogonal. Then we will normalize all the vectors, to obtain an orthonormal basis for $V$, $\{q_1, \ldots, q_k, q_{k+1}, \ldots, q_n\}$.

   Show that $\{q_{k+1}, \ldots, q_n\}$ are an orthonormal basis for $H^\perp$.

   (c) Find all the eigenpairs of $\mathcal{L}_H$ (You should be able to do this without resorting to the matrix of the transformation. Consider what’s invariant under $\mathcal{L}_H$.)

   (d) Based on your previous answer, would the matrix of the transformation $[\mathcal{L}_H]_Q$, where $Q = \{q_1, \ldots, q_n\}$ is the basis for $V$, be diagonalizable? Your previous answer should allow you to provide the answer here, however, if you don’t see it, form $[\mathcal{L}_H]_Q$ and see what you get, it’s neat!

4. Let $V = \mathbb{R}^n$. Let $\{b_1, \ldots, b_k\}$ be a set of linearly independent vectors. Let $\{w_1, \ldots, w_k\}$ be another set of linearly independent vectors. Assume that the vectors satisfy the following so-called “bi-orthogonality” condition$^1$:

   $b_i^T w_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

   Define $B = [b_1, \ldots, b_k]$ and $W = [w_1, \ldots, w_k]$ as $2 \times n \times k$ matrices. 

   Show that $P = BW^T$ is a projector (i.e. that $P$ is idempotent), explain why Range($P$) is $\text{span}\{b_1, \ldots, b_k\}$, then show it is not necessarily an orthogonal projector (i.e. we don’t necessary have $P^T = P$.)

$^1$It is possible to achieve such sets under the right conditions using Gram-Schmidt-like approach.