DIVISORIAL COMPONENTS OF THE PETRI LOCUS FOR PENCILS

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1. Introduction

Let $C$ be a generic curve of genus $g$ and $L$ a line bundle on $C$ of degree $d$ with $k$ sections. The Petri map is the natural cup-product map

$$H^0(C, L) \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K).$$

The target space of this map, $H^0(K)$ can be identified with the dual of the tangent space to the Picard Variety.

Denote by $W^r_d$ the locus of line bundles of degree $d$ with at least $k = r + 1$ sections. The orthogonal to the image of the Petri map is the tangent space to $W^r_d$ at the point $L$.

It is known that, on the generic curve and for every line bundle on it, the Petri map is injective. This was stated by Petri and first proved by Gieseker and later by Eisenbud and Harris using reducible curves. Some proofs that do not use reducible curves were given later. Denote by $\rho = g - (r + 1)(g - d + r)$, the Brill-Noether number that gives the expected dimension of $W^r_d$ and the minimum dimension of any of its components. Note that this number is the difference between the dimensions of the target space and the domain of the Petri map. Hence, if the Petri map is injective, $W^r_d$ is non-singular at $L$ and of the expected dimension.

The locus where the Petri map fails to be injective will be called the Petri locus. It is expected to be a divisor in the moduli space of curves $\mathcal{M}_g$ when the Brill-Noether number $\rho$ is positive. In fact, assume that the Brill-Noether number is positive. It is known then that every curve of genus $g$ has linear series of degree $d$ with $k$ sections and the generic curve has a scheme of dimension $\rho$ of such linear series. Consider a scheme (defined locally) parameterizing pairs consisting of a curve and one such linear series. It has dimension $3g - 3 + \rho$. The locus of line bundles where the Petri map fails to be injective may be constructed as a locally determinantal variety. The number of independent conditions imposed by the non-injectivity is expected to be $\rho + 1$. So, any component of the locus in the set of pairs of a curve and a line bundle
will have dimension at least $3g - 4$. In order to prove that there exists a divisorial component of the Petri locus in the moduli space of curves, it suffices then to exhibit a curve in the locus with only a finite number of linear series of this degree and number of sections for which the Petri condition fails. We shall do that here for the case of a pencil $(r = 1)$.

1.1. Theorem When $2d - g - 2 \geq 0$, the Petri locus for $r = 1$ has a divisorial component.

This result was also proved by Farkas in [F] with an inductive argument over the genus.

The structure of the paper is as follows. In the next section, we present a minor modification of the proof of Eisenbud and Harris of the injectivity of the Petri map on the generic curve. We replace their curves by a slightly more general type of curves (that do not require characteristic zero) (see [E,H2] and also [W]). We include this here because we need to use the arguments in this proof later. In section three, we produce some (special) reducible curves in the Petri locus. Then in section four we show that for a given degree, the curves in section three have only a finite number of linear series for which the Petri map fails to be injective.

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2. Proof of the Gieseker-Petri Theorem using chains of elliptic components

Fix a genus $g$ and the degree $d$ for line bundles $L$ on $C$. We shall denote by $k$ (rather than the classical $r + 1$) the dimension of the space of sections of these line bundles. We shall assume that $k, g, d$ have been fixed so that the generic curve of genus $g$ has line bundles of degree $d$ with $k$ sections (equivalently, the Brill- Noether number $g - k(g - 1 - d + k)$ is non-negative).

In order to prove the injectivity of the Petri map for a generic curve, it suffices to prove it for a special curve. Consider a family of curves $\pi : \mathcal{C} \to T$. Let $T$ be the spectrum of a discrete valuation ring $\mathcal{O}$ with maximal ideal generated by $t$. Assume that the generic fiber of $\pi$ is a non-singular curve and the special fiber $C$ looks as follows:

Take $g$ elliptic curves $E^i$ and let $P^i, Q^i$ be generic points on $E^i$. Take any number of rational curves $C_1^0, \ldots C_{k_0}^0, \ldots C_0^g, \ldots C_{k_g}^g$ again with points
Glue $C_{ij}$ to $C_{i+1j}$ by identifying $Q_{ij}$ to $P_{i+1j}$. Glue $C_{i-1k}$ to $E^i$ by identifying $Q_{i-1k}$ to $P^i$. Glue $E^i$ to $C_{i1}$ by identifying $Q_{i-1k}$ to $P_{i1}$.

For convenience of notation, we shall denote by $Y_1, \ldots, Y_M$, $M = k_0 + \ldots + k_g + g$ the components of $C$ starting with $C^0_1$ and ending with $C^g_{kg}$. We shall denote by $P_i, Q_i$ the two points in $Y_i$ that get identified to $Q_{i-1k} \in Y_{i-1}$ and $P_{i+1} \in Y_{i+1}$ respectively. We warn the reader that we shall keep the superindices $i$ when we need to refer to the $i^{th}$ elliptic curve. We hope this will not produce too much confusion.

Note that the form of the central fiber does not change if we make base changes and normalisations.

Consider now the set up of limit of a linear series in the sense of Eisenbud and Harris ([E,H2] p.273 or [E,H1], section 2). Let $L$ be a line bundle on $C$ of given degree $d$. One can modify $L$ by tensoring with a divisor with support on the central fiber. This leaves invariant the line bundle on the generic fiber but modifies it in the central fiber. For every component $Y_i$ of $C$, there is a line bundle $L_i$ on $C$ such that it has degree zero on every component of the central fiber except for the component $Y_i$. As $L_i = L_{i+1}(-d \sum_{j < i} Y_j)$, one can identify $L_i$ with a subsheaf of $L_{i+1}$.

Consider $\pi^*_i(L_i)$ This is a free $O$ module of rank $k$. Moreover, $L_i \subset L_{i+1}$ is a lattice. Denote by $V_i$ the image of the restriction map

$$\pi^*_i L_i \rightarrow \pi^*_i L_i|_{Y_i} = H^0(Y_i, L_i|_{Y_i})$$

As $deg L_i|_{Y_j} = 0, \ j \neq i$, this map is injective and we shall sometimes identify $\pi^*_i(L_i)$ with $V_i$.

**2.1.** One can find a basis of sections $\sigma^i_m, m = 1..k$ of the free module $\pi^*_i(L_i)$ such that the orders of vanishing of the sections $\sigma^i_m$ at $P_i$ are the different orders of vanishing of the sections of $V_i$ and $t^\alpha \sigma^i_m, \ m = 1..k$ form a basis for $\pi^*_i(L_{i+1})$.

For a proof see [E,H2] Lemma 1.2.

We shall now relate the vanishing of sections of line bundles at the various nodes.

**2.2. Lemma** 1) Let $Y$ be an irreducible non-singular curve $L$ a line bundle of degree $d$ on $Y$ and $P, Q$ two points on $Y$. The sum of the orders of vanishing at $P$ and $Q$ of any section of $L$ is at most $d$.

2) Let $Y$ be an elliptic curve and $P, Q$ generic points of $Y$. Let $L$ be a line bundle of degree $d$ on $Y$. The sum of the orders of vanishing
at $P$ and $Q$ of any section of $L$ is at most $d - 1$ except in the case where $L = \mathcal{O}(aP + (d - a)Q)$ for some $a$. In this case, there is only one section of $L$ vanishing with multiplicities adding up to $d$ at the two points.

**Proof.** If a section of a line bundle $L$ vanishes with order $a$ at $P$ and $b$ at $Q$, then $\mathcal{O}(aP + bQ)$ is a subsheaf of $L$. Hence, $d \geq a + b$. This proves the first statement.

If $a + b = d$, then $L = \mathcal{O}(aP + bQ)$. If there is another section vanishing to orders $a', b'$ at $P, Q$ with $a' + b' = d$ and say $a' > a$ then $aP + bQ \equiv a'P + b'Q$. Hence, $cP \equiv cQ$ with $c = a' - a$. This contradicts the genericity of the pair $P, Q$ if $Y$ is not rational. \hfill \Box

**Remark** The genericity of $P, Q$ is essential here. If $cP$ is linearly equivalent to $cQ$ for some $c \leq d$, then the line bundle $\mathcal{O}(aP + (d - a)Q)$, $a \geq c$ has (at least) two sections with orders of vanishing adding up to $d$ at $P, Q$ namely $aP + (d - a)Q$ and $(a - c)P + (d - a + c)Q$.

The following result of Eisenbud and Harris (cf. Prop 1.1 in [E,H2]), will be used in the sequel.

**2.3. Lemma** Let $\sigma$ be a section in $\pi_*\mathcal{L}_i$. Let $\alpha$ be the unique integer such that $t^\alpha\sigma \in \pi_*(\mathcal{L}_{i+1}) - t\pi_*(\mathcal{L}_{i+1})$, then

$$ord_{P_i}(\sigma|_{C_i}) \leq d - ord_{Q_i}(\sigma|_{C_i}) \leq \alpha \leq ord_{P_{i+1}}(t^\alpha\sigma|_{C_{i+1}})$$

Denote by $\mathcal{K}$ the canonical sheaf on $\pi : \mathcal{C} \to T$. There is a limit linear series of dimension $g$ associated to the canonical sheaf on the central fiber that we describe next:

**2.4. Lemma** The canonical limit linear series on $C$ has line bundles on $E^i$ equal to $K^i = \mathcal{O}(2(i - 1)P^i + 2(g - i)Q^i)$ while on the rational components the line bundle is $\mathcal{O}(2(g - 1))$. The space of sections on $E^i$ is $H^0(L_i(-(i - 2)P - 2g - 2i)Q) \oplus H^0(L_i(2(i - 1)P - (g - i - 1)Q))$. The unique section whose order of vanishing at $P$ and $Q$ is $2g - 2$ vanishes with order $2(i - 1)$ at $P$ and $2g - 2i$ at $Q$.

Consider now the limit linear series corresponding to $\mathcal{K} \otimes \mathcal{L}^{-1}$. Denote by $\sigma'$ its sections.

Let $\beta_i'$ be the unique integer such that $t^{\beta_i'}\sigma' \in \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1}) - t\pi_*(\mathcal{K} \otimes \mathcal{L}^{-1})$, then again a result as in 2.1 holds.

Consider now the Petri map

$$\pi_*\mathcal{L}_Y \otimes \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1}) \to \pi_*\mathcal{K}$$

As in [E,H2], p.277, one can define the order of a section $\rho \in \pi_*\mathcal{L}_Y \otimes \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1})$ at a point $P$ in a component $Y$ as follows:
2.5. Definition We say $\text{ord}_P(\rho_{\mathcal{Y}_i}) \geq l$ if and only if $\rho$ is in the linear span of $t(\pi_*\mathcal{L}_Y \otimes \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1}))$ and elements of the form $\sigma_m \otimes \sigma'_n$ where $\text{ord}_P(\sigma_m) + \text{ord}_P(\sigma'_n) \geq l$, $\sigma_m \in \pi_*(\mathcal{L}_Y)$, $\sigma'_n \in \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1})$.

One then has the following result (cf. [E,H2], Lemma 3.2)

2.6. Lemma Let $\sigma_m$ be a basis of the free $\mathcal{O}$ module $\pi_*(\mathcal{L}_i)$ such that the orders of vanishing of the $\sigma_m$ at $P_i$ are the distinct orders of vanishing of the linear series at this point and $t^\alpha \sigma_m$ is a basis of $\pi_*(\mathcal{L}_{i+1})$. Let $\sigma'_n$ be a basis of the free $\mathcal{O}$ module $\pi_*(\mathcal{K} \otimes \mathcal{L}^{-1}))$ such that the orders of vanishing of the $\sigma'_n$ at $P_i$ are the distinct orders of vanishing of the linear series at this point and $t^\alpha \sigma'_n$ is a basis of $\pi_*(\mathcal{K} \otimes \mathcal{L}^{-1}))_i$. If

$$\rho = \sum f_{n,m} (\sigma_m \otimes \sigma'_n)$$

where the $f_{n,m}$ are functions on the discrete valuation ring $\mathcal{O}$ and the associated discrete valuation is $\nu$, then

$$\text{ord}_P(\rho_{\mathcal{Y}_i}) = \min_{\nu(f_{n,m})=0} (\text{ord}_P(\sigma_m) + \text{ord}_P(\sigma'_n))$$

If $\beta$ is the unique integer such that

$$t^\beta \rho \in p_*\mathcal{L}_Y \otimes \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1})_{i+1} - t(p_*\mathcal{L}_Y \otimes \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1})_{i+1})$$

then

$$\beta = \max\{\alpha_n + \alpha_m - \nu(f_{n,m})\}$$

Let us assume now that the kernel of the Petri map is non-zero on the generic curve. We can then find an element $\rho$ such that say

$$\rho \in \pi_*(\mathcal{L}_Y \otimes \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1})_{C_{i+1}} - t(p_*\mathcal{L}_Y \otimes \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1})_{C_{i+1}})$$

and $t^\beta \rho \in \mathcal{K}r(\pi_*(\mathcal{L}_Y \otimes \pi_*(\mathcal{K} \otimes \mathcal{L}^{-1})_{C_{i+1}}) \to \pi_*(\mathcal{K}_{C_{i+1}})$.

As a section of a line bundle cannot vanish to order higher than the degree, the following claim will conclude the proof.

2.7. Claim If $l > k_0 + \ldots + k_{m-1} + m$, then $\text{ord}_{P_i}(t^\beta \rho) \geq 2m$. In particular, for $l \geq k_0 + \ldots + k_{g-1} + g$, $\text{ord}_{P_i}(t^\beta \rho) \geq 2g$.

Proof. We prove the following two statements:

1) If $C_i$ is a rational curve,

$$\text{ord}_{P_{i+1}}(t^{\beta_i+1} \rho_{\mathcal{Y}_{i+1}}) \geq \text{ord}_{P_i}(t^\beta \rho_{\mathcal{Y}_i})$$

2) If $C_i$ is an elliptic component,

$$\text{ord}_{P_{i+1}}(t^{\beta_i+1} \rho_{\mathcal{Y}_{i+1}}) \geq \text{ord}_{P_i}(t^\beta \rho_{\mathcal{Y}_i}) + 2$$

In other words, the order of vanishing of a section $t^\beta_i \rho$ at $P_{i+1}$ is at least as large as that of $t^\beta_i \rho$ at $P_i$ if $Y_i$ is rational and at least two
units larger if it is elliptic. As the order of vanishing $t^{\beta_0} \rho$ at $P_0$ is non-negative, this shows that the order of vanishing of $t^{\beta_0} \rho$ at $P_i$ is at least $2m$ if $m$ elliptic curves precede $Y_i$. This is the first part of the claim. The second part follows from the first when $m = g$.

We now turn to the proof of 1) and 2). Choose a basis $\sigma_m$, $m = 1 \ldots k$ of $\pi_*(\mathcal{L}_i)$ such that $t^{\alpha_m} \sigma_m$ is a basis of $\pi_*\mathcal{L}_{i+1}$. For simplicity of notation, we shall assume that $\beta_i = 0$. Similarly, choose a basis $\sigma'_n$, $n = 1 \ldots k' = k - d + g - 1$ of $\pi_*(\mathcal{K} \otimes \mathcal{L}^{-1}_i)$ such that $t^{\alpha_n} \sigma'_n$ is a basis of $\pi_*\mathcal{L}_{i+1}$. Write

$$\rho = \sum_{m,n} f_{m,n}(\sigma_m \otimes \sigma'_n)$$

Then, from 2.6,

$$\text{ord}_{P_i} (\rho_{Y_i}) = \min_{\nu(f_{m,n})=0} \left( \text{ord}_{P_i} (\sigma_m) + \text{ord}_{P_i} (\sigma'_n) \right)$$

Assume that this minimum is attained by a pair corresponding to the indices $m_0, n_0$ with $\nu(f_{m_0,n_0}) = 0$. Then from 2.3,

$$(\text{ord}_{P_i} (\sigma_{m_0}) + \text{ord}_{P_i} (\sigma'_{n_0})) \leq 2d - \text{ord}_{Q_i} (\sigma_{m_0}) - \text{ord}_{Q_i} (\sigma'_{n_0}) \leq \alpha_{m_0} + \alpha'_{n_0}$$

From 2.6 and the fact that $\nu(f_{m_0,n_0}) = 0$, the latter is at most $\beta_{i+1}$.

Write

$$t^{\beta_{i+1}} \rho = \sum_{n \leq m} (t^{\beta_{i+1} - \alpha_m - \alpha'_n} f_{nm})(t^{\alpha_m} \sigma_m \otimes t^{\alpha_n} \sigma'_n)$$

Hence, from 2.6

$$\text{ord}_{P_{i+1}} (t^{\beta_{i+1}} \rho_{Y_{i+1}}) = \min_{\beta_{i+1} - \alpha_m - \alpha'_n + \nu(f_{nm})=0} \left( \text{ord}_{P_{i+1}} (t^{\alpha_m} \sigma_m) + \text{ord}_{P_{i+1}} (t^{\alpha_n} \sigma'_n) \right)$$

Assume that this minimum is attained at a pair $m_1, n_1$ with

$$\beta_{i+1} - \alpha_{m_1} - \alpha'_{n_1} + \nu(f_{m_1,n_1}) = 0$$

Then,

$$\beta_{i+1} \leq \beta_{i+1} + \nu(f_{m_1,n_1}) = \alpha_{m_1} + \alpha'_{n_1} \leq \text{ord}_{P_{i+1}} (t^{\alpha_{m_1}} \sigma_{m_1}) + \text{ord}_{P_{i+1}} (t^{\alpha'_{n_1}} \sigma'_{n_1})$$

where the last inequality comes from 2.3

Stringing together the above inequalities, we obtain

$$\text{ord}_{P_i} (\rho_{Y_i}) \leq \text{ord}_{P_{i+1}} (\rho_{Y_{i+1}}).$$

Hence part 1) is proved.

Assume now that there is equality in the inequality above. Then all the previous inequalities must be equalities. In particular, any terms $\sigma_m \otimes \sigma'_n$ that give the vanishing of $\rho$ at $P_i$ satisfy

$$\text{ord}_{P_i} (\sigma_m) + \text{ord}_{Q_i} (\sigma_m) = d, \text{ord}_{P_i} (\sigma'_m) + \text{ord}_{Q_i} (\sigma'_n) = 2g - 2 - d.$$
not in the kernel of the Petri map, the vanishing must go up by at least one. Let us check that it goes up by two.

If we have \( \operatorname{ord}_{Y+1}(\rho|_{Y+1}) = \operatorname{ord}_{Y}(\rho|_{Y}) + 1 \), then in each pair that gives the vanishing of \( \rho \) at \( P_i \), one of \( \sigma_m, \sigma'_m \) would vanish to order \( d, 2g-2-d \) respectively between the two nodes. From 2.2, there is at most one such section \( \sigma_{i_0}, \sigma'_{i_0} \) for the restrictions to the elliptic curve of \( \mathcal{L} \) and \( \mathcal{K} \otimes \mathcal{L}^{-1} \) respectively.

Hence, if \( \operatorname{ord}_{Y+1}(\rho|_{Y+1}) \leq \operatorname{ord}_{Y}(\rho|_{Y}) + 1 \), the terms in \( \rho \) giving the vanishing at \( P_i \) could be written as

\[
\sigma_{i_0} \otimes \sigma' + \sigma \otimes \sigma'_{i_0}
\]

for some sections \( \sigma, \sigma' \).

Assume now that \( C_i = E^m \) is the \( m \)-th elliptic curve and that 2) has been proved for all elliptic components preceding \( C_0 \). As 1) has already been proved, we have

\[
\operatorname{ord}_{P_i}(\rho) \geq 2m - 2
\]

We want to show that \( \operatorname{ord}_{P_{i+1}}(t^a \rho) \geq 2m \). From the discussion above, the result will follow except in the case that the only terms that give the vanishing at \( \rho \) are \( \sigma_{i_0} \otimes \sigma' + \sigma \otimes \sigma'_{i_0} \). Here \( \sigma_{i_0}, \sigma'_{i_0} \) are again the unique sections of \( \mathcal{L} \) and \( \mathcal{K} \otimes \mathcal{L}^{-1} \) that vanish only at \( P_i, Q_i \). Hence, \( \sigma_{i_0} \sigma'_{i_0} \) is a section of the canonical that vanishes only at \( P_i, Q_i \). It follows from 2.4 that it vanishes at \( P_i \) to order \( 2(m-1) \). Then

\[
2m-2 \leq \operatorname{ord}_{P_i}(\rho) = \operatorname{ord}_{P_i}(\sigma_{i_0}) + \operatorname{ord}_{P_i}(\sigma') \neq \operatorname{ord}_{P_i}(\sigma_{i_0}) + \operatorname{ord}_{P_i}(\sigma'_{i_0}) = 2m-2.
\]

Hence, \( \operatorname{ord}_{P_i}(\sigma_{i_0}) + \operatorname{ord}_{P_i}(\sigma') \geq 2m - 1 \).

From 2.3,

\[
\operatorname{ord}_{P_i}(\sigma') \leq 2g - 2 - d - \operatorname{ord}_{Q_i}(\sigma') \leq \alpha' \leq \operatorname{ord}_{P_{i+1}}(t^a \sigma').
\]

The first inequality is strict. Hence,

\[
\operatorname{ord}_{P_{i+1}}(t^a \rho) = \operatorname{ord}_{P_{i+1}}(t^a \sigma_{i_0}) + \operatorname{ord}_{P_{i+1}}(t^a \sigma') \geq \operatorname{ord}_{P_i}(\sigma_{i_0}) + \operatorname{ord}_{P_i}(\sigma') + 1 \geq 2m - 1 + 1 = 2m.
\]

This completes the proof

\[ \square \]

3. A curve in the Petri locus

In this section, we shall display special curves when \( k = 2 \) for which the Petri map is not injective. We start with the case of odd \( g \). Our special curve will be as described above except that on the \( g-1 \)-th elliptic curve the two points are no longer generic but are assumed to satisfy the condition \( 2P = 2Q \) (or equivalently, \( P - Q \) is a torsion point
of order two on the elliptic curve). We now display a limit linear series on the curve with two sections such that the kernel of the Petri map is non zero.

Assume first that $\alpha = \frac{3g-3}{2} - d$ is even and write $\alpha = \frac{3g-3}{2} - d = 2i_0$. On the elliptic curve $E^j$, $j \leq 2i_0$, the restriction of the limit linear series will be taken to be

$$|2P| + (i - 1)P + (d - i - 1)Q, \ j = 2i$$

$$|2Q| + iP + (d - i - 2)Q, \ j = 2i + 1.$$  

The residual line bundle is then

$$(3i - 3)P + (2g - d - 3i + 1)Q, \ j = 2i$$

$$3iP + (2g - d - 3i - 2)Q, \ j = 2i + 1.$$  

For even $j = 2i$, denote by $t^j_{d-i-1} t^j_{2g-d-3i-2} t^{3(i-1)}_{2g-d-3i-2} s_0^i$ a section of $(3i - 5)P + (2g - d - 3i + 1)Q$ vanishing with order $3(i - 1)$ at $P$ and $2g - d - 3i - 2$ at $Q$. Let $s_0^i$ be a section of $|2P|$ vanishing to order two at $P$ and $s_1^i$ be a section of $|2P|$ vanishing to order one at $Q$.

Then the restriction to $E^{2i}$ of the section $\sigma$ in the kernel of the Petri map will be taken to be

$$t^i_{d-i-1} s_0^1 \otimes t^3_{2g-d-3i-2} s_1^0 - t^i_{d-i-1} s_1^0 \otimes t^{3(i-1)}_{2g-d-3i-2} s_0^i.$$  

For odd $j = 2i + 1$, denote by $t^i_{d-i-2}$ a section of $iP + (d - i - 2)Q$ vanishing with order $i$ at $P$ and $d - i - 2$ at $Q$. Denote by $t^i_{2g-d-3i-4}$ a section of $3iP + (2g - d - 3i - 4)Q$ vanishing with order $3i - 1$ at $P$ and $2g - d - 3i - 4$ at $Q$. Let $s_0^i$ be a section of $|2Q|$ vanishing to order one at $P$ and $s_0^i$ be a section of $|2Q|$ vanishing to order two at $Q$.

Then the restriction to $E^{2i+1}$ of the section $\sigma$ in the kernel of the Petri map will be taken to be

$$t^i_{d-i-2} s_0^1 \otimes t^3_{2g-d-3i-4} s_0^i - t^i_{d-i-2} s_0^i \otimes t^{3i-1}_{2g-d-3i-4} s_0^1.$$  

For even $j = 2(i_0 + k) \leq g - 3$, $k \geq 1$, take the restriction of the line bundle to be

$$\mathcal{O}((i_0 + 3k - 2)P + (d - i_0 - 3k + 2)Q).$$  

Then the residual line bundle is

$$\mathcal{O}((3i_0 + k)P + (2g - d - 3i_0 - k - 2)Q).$$  

Denote by $t^i_{d-i-3k-4}$ a section of $\mathcal{O}(i_0 + 3k - 4)P + (d - i_0 - 3k + 2)Q$ vanishing to order $i_0 + 3k - 2$ at $P$ and $d - i_0 - 3k - 1$ at $Q$. Denote by $t^{i_0+k-2}_{2g-d-3i_0-k-2}$ a section of $(3i_0 + k - 2)P + (2g - d - 3i_0 - k - 2)Q$ vanishing with order $3i_0 + k - 2$ at $P$ and $2g - d - 3i_0 - k - 2$ at $Q$. Let
$s_0^2$ be a section of $|2P|$ vanishing to order two at $P$ and $s_1^0$ be a section of $|2P|$ vanishing to order one at $Q$.

Then the restriction to $E^{i_0+k}$ of the section in the kernel of the Petri map will be taken to be

$$t_{d-i_0-3k-1}^i \otimes \tilde{E}^{i_0+k-2} - t_{d-i_0-3k-1}^i \otimes \tilde{E}^{i_0+k-2}$$

For odd $j = 2(i_0 + k) + 1 \leq g - 2$, take the restriction of the line bundle to be

$$\mathcal{O}((i_0 + 3k + 2)P + (d - i_0 - 3k - 2)Q)$$

Then the residual line bundle is

$$\mathcal{O}((3i_0 + k - 2)P + (2g - d - 3i_0 - k)Q).$$

Denote by $t_{d-i_0-3k-3}^{i_0+3k}$ a section of $(i_0 + 3k + 2)P + (d - i_0 - 3k - 2)Q$ vanishing with order $i_0 + 3k$ at $P$ and $d - i_0 - 3k - 3$ at $Q$. Denote by $\tilde{E}^{i_0+k-2}$ a section of $(3i_0 + k - 2)P + (2g - d - 3i_0 - k - 2)Q$ vanishing with order $3i_0 + k - 2$ at $P$ and $2g - d - 3i_0 - k - 2$ at $Q$. Let $s_0^1$ be a section of $|2Q|$ vanishing to order one at $P$ and $s_2^0$ be a section of $|2Q|$ vanishing to order two at $Q$.

Then the restriction to $E^{(i_0+k)+1}$ of the section in the kernel of the Petri map will be taken to be

$$t_{d-i_0-3k-3}^{i_0+3k} \otimes \tilde{E}^{i_0+k-2} - t_{d-i_0-3k-3}^{i_0+3k} \otimes \tilde{E}^{i_0+k-2}$$

On $E^{d-1}$, take the line bundle to be $\mathcal{O}(dP) = \mathcal{O}((d - 2)P + 2Q))$.

Denote by $t_0^{d-2}$ a section of $(d - 2)P$ vanishing with order $d - 2$ at $P$. Denote by $\tilde{E}^{2g-d-4}$ a section of $(2g - d - 4)Q$ vanishing to order $2g - d - 4$ at $P$. Let $s_2^2$ be a section of $|2P| = |2Q|$ vanishing to order two at $P$ and $s_2^0$ be a section of $|2P|$ vanishing to order two at $Q$.

Then the restriction to $E^{g-1}$ of the section in the kernel of the Petri map will be taken to be

$$t_0^{d-2} \otimes \tilde{E}^{2g-d-4} - t_0^{d-2} \otimes \tilde{E}^{2g-d-4}$$

On $E^g$, take the line bundle to be $\mathcal{O}(dP)$. Then the residual line bundle is $\mathcal{O}((2g - d)P)$.

Denote by $t_0^{d-2}$ a section of $(d - 2)P$ vanishing with order $d - 2$ at $P$. Denote by $\tilde{E}^{2g-d-4}$ a section of $(2g - d - 4)Q$ vanishing to order $2g - d - 4$ at $Q$. Let $s_0^2$ be a section of $|2P|$ vanishing to order two at $P$ and $s_1^0$ be a section of $|2P|$ vanishing to order one at $Q$.

Then the restriction to $E^g$ of the section in the kernel of the Petri map will be taken to be

$$t_0^{d-2} \otimes \tilde{E}^{2g-d-4} - t_0^{d-2} \otimes \tilde{E}^{2g-d-4}$$
If $\alpha = \frac{3g-3}{2} - d$ is odd, write $\alpha = \frac{3g-3}{2} - d = 2i_0 + 1$. On the elliptic curve $E^j$, $j \leq 2i_0 + 1$, the restriction of the limit linear series and the element in the kernel of the Petri map will be taken to be exactly as before.

For even $j = 2(i_0 + k) \leq g - 3$, $k \geq 1$, take the restriction of the line bundle to be

$$\mathcal{O}((i_0 + 3k - 3)P + (d - i_0 - 3k + 3)Q).$$

Then the residual line bundle is

$$\mathcal{O}((3i_0 + k + 1)P + (2g - d - 3i_0 - 3k - 3)Q).$$

Denote by $t_{i_0 + 3k - 3}^{0}$ a section of $\mathcal{O}(i_0 + 3k - 4)P + (d - i_0 - 3k + 2)Q$ vanishing to order $i_0 + 3k - 3$ at $P$ and $d - i_0 - 3k$ at $Q$. Denote by $t_{2g - d - 3i_0 - k - 3}^{0}$ a section of $(3i_0 + k - 1)P + (2g - d - 3i_0 - k - 3)Q$ vanishing with order $3i_0 + k - 1$ at $P$ and $2g - d - 3i_0 - k - 3$ at $Q$. Let $s_1^2$ be a section of $|2P|$ vanishing to order two at $P$ and $s_1^0$ be a section of $|2P|$ vanishing to order one at $Q$.

Then the restriction to $E^{2(i_0 + k)}$ of the section in the kernel of the Petri map will be taken to be

$$t_{i_0 + 3k - 3}^{0} \otimes t_{2g - d - 3i_0 - k - 3}^{0} - t_{i_0 + 3k - 3}^{0} \otimes t_{2g - d - 3i_0 - k - 3}^{0}.$$

For odd $j = 2(i_0 + k) + 1$, take the restriction of the line bundle to be

$$\mathcal{O}((i_0 + 3k)P + (d - i_0 - 3k - 2)Q).$$

Then the residual line bundle is

$$\mathcal{O}((3i_0 + k)P + (2g - d - 3i_0 - k - 3)Q).$$

Denote by $t_{i_0 + 3k - 1}^{0}$ a section of $(i_0 + 3k)P + (d - i_0 - 3k - 2)Q$ vanishing to order $i_0 + 3k$ at $P$ and $d - i_0 - 3k - 1$ at $Q$. Denote by $t_{2g - d - 3i_0 - k - 4}^{0}$ a section of $(3i_0 + k)P + (2g - d - 3i_0 - k - 4)Q$ vanishing with order $3i_0 + k$ at $P$ and $2g - d - 3i_0 - k - 4$ at $Q$. Let $s_1^0$ be a section of $|2Q|$ vanishing to order one at $P$ and $s_2^0$ be a section of $|2Q|$ vanishing to order two at $Q$.

Then the restriction to $E^{2(i_0 + k) + 1}$ of the section in the kernel of the Petri map will be taken to be

$$t_{i_0 + 3k - 1}^{0} \otimes t_{2g - d - 3i_0 - k - 4}^{0} - t_{i_0 + 3k - 1}^{0} \otimes t_{2g - d - 3i_0 - k - 4}^{0}.$$

On the curve $E^g - 1, E^g$, take the same as in the previous case.

Assume now that $g$ is even. Take the curve as in section one except that the points $P, Q$ in $E^{g-2}$ satisfy $2P = 2Q$.

Assume first that $\alpha' = \frac{3g}{2} - d - 2$ is even and write $\alpha' = \frac{3g}{2} - d - 2 = 2i_0$.

The description of the linear series and element in the kernel of the Petri
map on \( C \), \( i \leq g - 3 \) is taken the same as as in the case of odd genus and even \( \alpha \) replacing the \( \alpha \) there by the new \( \alpha' \) and changing the value of \( i_0 \) accordingly.

On \( E^{g-2} \), take the line bundle to be \( \mathcal{O}((d - 3)P + 3Q) = \mathcal{O}((d - 1)P + Q) \).

Denote by \( t_1^{d-3} \) a section of \( (d - 3)P + Q \) vanishing with order \( d - 3 \) at \( P \) and one at \( Q \). Denote by \( t_1^{2g-d-5} \) a section of \( (2g - d - 5)P + Q \) vanishing to order \( 2g - d - 5 \) at \( P \) and one at \( Q \). Let \( s_0^2 \) be a section of \( |2P| = |2Q| \) vanishing to order two at \( P \) and \( s_0^2 \) be a section of \( |2P| \) vanishing to order two at \( Q \).

Then the restriction to \( E^{g-2} \) of the section in the kernel of the Petri map will be taken to be

\[
\frac{t_1^{d-3} s_0^2 \otimes t_1^{2g-d-5} s_0^2}{s_2^2 - t_1^{d-3} s_1^0 \otimes t_1^{2g-d-5} s_0^2}
\]

On \( E^{g-1} \), take the line bundle to be \( \mathcal{O}((d - 1)P + Q) \).

Denote by \( t_1^{d-3} \) a section of \( (d - 3)P + Q \) vanishing with order \( d - 3 \) at \( P \) and one at \( Q \). Denote by \( t_1^{2g-d-5} \) a section of \( (2g - d - 5)P + Q \) vanishing to order \( 2g - d - 5 \) at \( P \) and one at \( Q \). Let \( s_0^2 \) be a section of \( |2P| \) vanishing to order two at \( P \) and \( s_1^0 \) a section vanishing to order one at \( Q \).

Then the restriction to \( E^{g-1} \) of the section in the kernel of the Petri map will be taken to be

\[
\frac{t_1^{d-3} s_0^2 \otimes t_1^{2g-d-5} s_0^2}{s_1^0 - t_1^{d-3} s_1^0 \otimes t_1^{2g-d-5} s_0^2}
\]

On \( E^g \), take the line bundle to be \( \mathcal{O}((d - 1)P + Q) \) where \( Q \) is a point such that \( 2P = 2Q \). Then the residual line bundle is \( \mathcal{O}((2g - 3 - d)P + Q) \).

Denote by \( t_0^{d-2} \) a section of \( (d - 2)P \) vanishing with order \( d - 2 \) at \( P \). Denote by \( t_0^{2g-d-4} \) a section of \( (2g - d - 4)Q \) vanishing to order \( 2g - d - 4 \) at \( Q \). Let \( s_1^1 \) be a section of \( |P + Q| \) vanishing to order one at \( P \) and \( s_0^0 \) be any other section of \( |P + Q| \).

Then the restriction to \( E^g \) of the section in the kernel of the Petri map will be taken to be

\[
\frac{t_0^{d-2} s_1^1 \otimes t_0^{2g-d-4} s_0^0}{t_0^{d-2} s_0^0 \otimes t_0^{2g-d-4} s_1^1}
\]

Assume now that \( g \) is even and that \( \alpha' = \frac{3g}{2} - d - 2 \) is odd and write \( \alpha' = \frac{3g}{2} - d - 2 = 2i_0 + 1 \). The description of the linear series and element in the kernel of the Petri map on \( C \), \( i \leq g - 3 \) is taken the same as as in the case of odd genus and odd \( \alpha \) replacing the \( \alpha \) there by the new \( \alpha' \) and changing the value of \( i_0 \) accordingly. For \( i \geq g - 2 \), the description is identical to the one in the case of \( \alpha' \) even.
4. Finiteness of the linear series in the Petri locus

We want to show that the curve $C$ of the previous section has only a finite number of limit linear series of degree $d$ with non-trivial kernel of the Petri map. We shall do so only in the case of odd genus $g$, the even genus case being similar.

Assume that there is a pencil of degree $d$ for which the kernel of the Petri map is not injective. Consider the aspect of the linear series on the curve $C_i$. Take a basis $a^i, b^i$ for the aspect on $C_i$ of the linear series and write the element in the kernel as $a^i \otimes \bar{a}^i + b^i \otimes \bar{b}^i$. Let $F$ be the fixed part of the series $< a^i, b^i >$ and $t_F$ a section corresponding to $F$. Hence, $a^i = t_F s^i_a, b^i = t_F s^i_b$. By the base-point-free pencil trick, one can write $\bar{a}^i = \bar{t}_F s^i_b, \bar{b}^i = -\bar{t}_F s^i_a$. We shall denote by $f, \bar{f}, d'$ the degrees of $F, \bar{F}$ and the fixed point free part of the linear series so that $d = d' + f, 2g - 2 - d = d' + \bar{f}$.

We recall that the sum of vanishings of a section of a line bundle at the points $P_i, Q_i$ is at most the degree of the line bundle and that when equality occurs, the line bundle must be of the form $\mathcal{O}(\lambda P + \mu Q)$.

From section one and the genericity of the curves $C_i, 1 \leq i \leq g - 2$, it follows that the vanishing of the section in the kernel of the Petri map is at least $2i - 2$ at $P_i$. Therefore, the vanishing of such a section at $P_{g-1}$ is at least $2g - 4$. It follows then that the vanishing at $Q_{g-1}$ is at most two.

The vanishing at $P_y$ of $s^y_a, s^y_b$ is (up to reordering) at most $d', d' - 2$ while the vanishing of the fixed parts $F^y, \bar{F}^y$ is at most $f^y, \bar{f}^y$. Then, as the vanishing at $Q_{g-1}$ is at most two, all these bounds must be attained and all the line bundles appearing on $C_g$ are special.

It follows then, that all the previous bounds are attained, hence the bundles on $C_{g-1}$ is also special.

On the previous curves (up to the $(g - 2)^{th}$), the vanishing for the section in the kernel goes down by two at each step. Hence

$$ord_P t^i_F + ord_Q t^i_F + ord_P \bar{t}^i_F + ord_Q \bar{t}^i_F +$$

$$+ ord_P s^i_a + ord_Q s^i_a + ord_P s^i_b + ord_Q s^i_b = 2g - 4$$

It follows that for at least two of the sections $\beta$ with $\beta$ one of the four $t^i, \bar{t}^i, s^i_a, s^i_b$, there is equality between

$$ord_P \beta + ord_Q \beta$$

and the degree of the line bundle that corresponds to $\beta$. Moreover, equality cannot occur for both $s^i_a, s^i_b$. If equality occurs for $t^i_F$ and one of the $s^i$, then the line bundle $L^i$ is special. Similarly if it occurs for $\bar{t}^i_F$ and one of the $s^i$, then the line bundle $K \otimes (L^i)^{-1}$ is special and
therefore $L^i$ is special as the restriction of $K$ is too. Finally if equality occurs for $t^i, \bar{t}^i$, then $F, \bar{F}$ are special and so is $K(-F - \bar{F})$ which is the square of the line bundle corresponding to the sections $s^i$. Hence this one is also special. It follows that the restriction of the line bundles to each curve are special and hence there is only a finite number of them.

\textbf{References}


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